

Cuspidal representations in the ℓ -adic cohomology of the Rapoport-Zink space for $\mathrm{GSp}(4)$

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ABSTRACT. In this paper, we study the ℓ -adic cohomology of the Rapoport-Zink tower for $\mathrm{GSp}(4)$. We prove that the smooth representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ obtained as the i th compactly supported ℓ -adic cohomology of the Rapoport-Zink tower has no quasi-cuspidal subquotient unless $i = 2, 3, 4$. Our proof is purely local and does not require global automorphic methods.

1 Introduction

In [RZ96], M. Rapoport and Th. Zink introduced certain moduli spaces of quasi-isogenies of p -divisible groups with additional structures called the *Rapoport-Zink spaces*. They constructed systems of rigid analytic coverings of them which we call the *Rapoport-Zink towers*, and established the p -adic uniformization theory of Shimura varieties generalizing classical Čerednik-Drinfeld uniformization. These spaces uniformize the rigid spaces associated with the formal completion of certain Shimura varieties along Newton strata.

Using the ℓ -adic cohomology of the Rapoport-Zink tower, we can construct a representation of the product $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, where G is the reductive group over \mathbb{Q}_p corresponding to the Shimura datum, J is an inner form of it, and $W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is the Weil group of the p -adic field \mathbb{Q}_p . It is widely believed that this realizes the local Langlands and Jacquet-Langlands correspondences (*cf.* [Rap95]). Classical examples of the Rapoport-Zink spaces are the Lubin-Tate space and the Drinfeld upper half space; these spaces were extensively studied by many people and many important results were obtained (*cf.* [Dri76], [Car90], [Har97], [HT01], [Dat07], [Boy09] and references therein). However, very little was known about the ℓ -adic cohomology of other Rapoport-Zink spaces.

The aim of this paper is to study cuspidal representations in the ℓ -adic cohomology of the Rapoport-Zink tower for $\mathrm{GSp}_4(\mathbb{Q}_p)$. Let us denote the Rapoport-Zink space for $\mathrm{GSp}_4(\mathbb{Q}_p)$ by $\check{\mathcal{M}}$. It is a special formal scheme over $\mathbb{Z}_{p^\infty} = W(\overline{\mathbb{F}_p})$ in the sense of Berkovich [Berk96]. Let $\check{\mathcal{M}}^{\mathrm{rig}}$ be the Raynaud generic fiber of $\check{\mathcal{M}}$, that is, the generic fiber of the adic space $t(\check{\mathcal{M}})$ associated with $\check{\mathcal{M}}$. Using level structures

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at p , we can construct the Rapoport-Zink tower

$$\cdots \longrightarrow \check{\mathcal{M}}_{m+1}^{\text{rig}} \longrightarrow \check{\mathcal{M}}_m^{\text{rig}} \longrightarrow \cdots \longrightarrow \check{\mathcal{M}}_2^{\text{rig}} \longrightarrow \check{\mathcal{M}}_1^{\text{rig}} \longrightarrow \check{\mathcal{M}}_0^{\text{rig}} = \check{\mathcal{M}}^{\text{rig}},$$

where $\check{\mathcal{M}}_m^{\text{rig}} \longrightarrow \check{\mathcal{M}}^{\text{rig}}$ is an étale Galois covering of rigid spaces with Galois group $\text{GSp}_4(\mathbb{Z}/p^m\mathbb{Z})$. We take the compactly supported ℓ -adic cohomology (in the sense of [Hub98]) and take the inductive limit of them. Then, on

$$H_{\text{RZ}}^i := \varinjlim_m H_c^i(\check{\mathcal{M}}_m^{\text{rig}} \otimes_{\mathbb{Q}_p^\infty} \overline{\mathbb{Q}_p^\infty}, \mathbb{Q}_\ell)$$

(here $\mathbb{Q}_p^\infty = \text{Frac } \mathbb{Z}_p^\infty$), we have an action of a product

$$\text{GSp}_4(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p),$$

where J is an inner form of GSp_4 .

The main theorem of this paper is as follows:

Theorem 1.1 (Theorem 3.2) *The $\text{GSp}_4(\mathbb{Q}_p)$ -representation $H_{\text{RZ}}^i \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell}$ has no quasi-cuspidal subquotient unless $i = 2, 3, 4$.*

For the definition of quasi-cuspidal representations, see [Bern84, 1.20]. Note that since $\check{\mathcal{M}}_m^{\text{rig}}$ is 3-dimensional for every $m \geq 0$, $H_{\text{RZ}}^i = 0$ unless $0 \leq i \leq 6$.

Our proof of this theorem is purely local. We do not use global automorphic methods. The main strategy of the proof is similar to that of [Mie10a], in which the analogous result for the Lubin-Tate tower is given; we construct the formal model $\check{\mathcal{M}}_m$ of $\check{\mathcal{M}}_m^{\text{rig}}$ by using Drinfeld level structures and consider the geometry of its special fiber. However, our situation is much more difficult than the case of the Lubin-Tate tower. In the Lubin-Tate case, the tower consists of affine formal schemes $\{\text{Spf } A_m\}_{m \geq 0}$, and we can associate it with the tower of affine schemes $\{\text{Spec } A_m\}_{m \geq 0}$. In [Mie10a], the second author defined the stratification on the special fiber of $\text{Spec } A_m$ by using the kernel of the universal Drinfeld level structure, and considered the local cohomology of the nearby cycle complex $R\psi\Lambda$ along the strata. On the other hand, our tower $\{\check{\mathcal{M}}_m\}_{m \geq 0}$ does not consist of affine formal schemes and there is no canonical way to associate it with a tower of schemes. To overcome this problem, we take a sheaf-theoretic approach. For each direct summand I of $(\mathbb{Z}/p^m\mathbb{Z})^4$, we will define the complex of sheaves $\mathcal{F}_{m,I}$ on $(\check{\mathcal{M}}_m)_{\text{red}}$ so that the cohomology $H^i((\check{\mathcal{M}}_m)_{\text{red}}, \mathcal{F}_{m,I})$ substitutes for the local cohomology of $R\psi\Lambda$ along the strata defined by I in the Lubin-Tate case. For the definition of $\mathcal{F}_{m,I}$, we use the p -adic uniformization theorem by Rapoport and Zink.

There is another difficulty; since a connected component of $\check{\mathcal{M}}$ is not quasi-compact, the representation H_{RZ}^i of $\text{GSp}_4(\mathbb{Q}_p)$ is far from admissible. Therefore it is important to consider the action of $J(\mathbb{Q}_p)$ on H_{RZ}^i , though it does not appear in our main theorem. However, the cohomology $H^i((\check{\mathcal{M}}_m)_{\text{red}}, \mathcal{F}_{m,I})$ has no apparent action of $J(\mathbb{Q}_p)$, since $J(\mathbb{Q}_p)$ does not act on the Shimura variety uniformized by

$\check{\mathcal{M}}$. We use the variants of formal nearby cycle introduced by the second author in [Mie10b] to endow it with an action of $J(\mathbb{Q}_p)$. Furthermore, to ensure the smoothness of this action, we use a property of finitely generated pro- p groups (Section 2). In fact, extensive use of the formalism developed in [Mie10b] make us possible to work mainly on the Rapoport-Zink tower itself and avoid the theory of p -adic uniformization except for proving that $\check{\mathcal{M}}_m$ is locally algebraizable. However, for the reader's convenience, we decided to make this article as independent of [Mie10b] as possible.

The authors expect that the converse of Theorem 1.1 also holds. Namely, we expect that $H_{\mathrm{RZ}}^i \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell}$ has a quasi-cuspidal subquotient if $i = 2, 3, 4$. We hope to investigate it in a future work.

The outline of this paper is as follows. In Section 2, we prepare a criterion for the smoothness of representations over \mathbb{Q}_ℓ . It is elementary but very powerful for our purpose. In Section 3, we give some basic definitions concerning with the Rapoport-Zink space for $\mathrm{GSp}(4)$ and state the main theorem. Section 4 is devoted to introduce certain Shimura varieties related to our Rapoport-Zink tower and recall the theory of p -adic uniformization. The proof of the main theorem is accomplished in Section 5. The final Section 6 is an appendix on cohomological correspondences. The results in the section are used to define actions of $\mathrm{GSp}_4(\mathbb{Q}_p)$ on various cohomology groups.

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Notation Let p be a prime number and take another prime ℓ with $\ell \neq p$. We denote the completion of the maximal unramified extension of \mathbb{Z}_p by \mathbb{Z}_{p^∞} and its fraction field by \mathbb{Q}_{p^∞} . Let $\mathbf{Nilp} = \mathbf{Nilp}_{\mathbb{Z}_{p^\infty}}$ be the category of \mathbb{Z}_{p^∞} -schemes on which p is locally nilpotent. For an object S of \mathbf{Nilp} , we put $\overline{S} = S \otimes_{\mathbb{Z}_{p^\infty}} \overline{\mathbb{F}_p}$.

In this paper, we use the theory of adic spaces ([Hub94], [Hub96]) as a framework of rigid geometry. A rigid space over \mathbb{Q}_{p^∞} is understood as an adic space locally of finite type over $\mathrm{Spa}(\mathbb{Q}_{p^\infty}, \mathbb{Z}_{p^\infty})$.

Every sheaf and cohomology are considered in the étale topology. Every smooth representation is considered over \mathbb{Q}_ℓ or $\overline{\mathbb{Q}_\ell}$. For a \mathbb{Q}_ℓ -vector space V , we put $V_{\overline{\mathbb{Q}_\ell}} = V \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell}$.

2 Preliminaries: smoothness of representations of profinite groups

Let \mathbf{G} be a linear algebraic group over a p -adic field F . In this section, we give a convenient criterion for the smoothness of a $\mathbf{G}(F)$ -representation over \mathbb{Q}_ℓ . The following theorem is essential:

Theorem 2.1 *Let K be a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ and (π, V) a finite-dimensional representation over \mathbb{Q}_ℓ of K as an abstract group. Assume that there exists a K -stable \mathbb{Z}_ℓ -lattice Λ of V . Then this representation is automatically smooth.*

In order to prove this theorem, we require several facts on pro- p groups. Put $K_1 = K \cap (1 + pM_n(\mathbb{Z}_p))$, which is a pro- p open subgroup of K .

Lemma 2.2 *The pro- p group K_1 is (topologically) finitely generated.*

Proof. By [DdSMS99, §5.1], the profinite group $GL_n(\mathbb{Z}_p)$ has finite rank. In particular, K_1 , a closed subgroup of $GL_n(\mathbb{Z}_p)$, has finite topological generators. \blacksquare

Lemma 2.3 *Every subgroup of finite index of K_1 is open.*

Proof. In fact, this is true for every finitely generated pro- p group; this is due to Serre [Ser94, 4.2, Exercices 6)]. See also [DdSMS99, Theorem 1.17], which gives a complete proof. \blacksquare

Remark 2.4 More generally, every subgroup of finite index of a finitely generated profinite group is open ([NS03], [NS07a], [NS07b]). It is a very deep theorem.

Lemma 2.5 *Let G be a pro- ℓ group. Then every homomorphism $f: K_1 \rightarrow G$ is trivial.*

Proof. Let H be an open normal subgroup of G and denote the composite $K_1 \xrightarrow{f} G \rightarrow G/H$ by f_H . By Lemma 2.3, $\text{Ker } f_H$ is an open normal subgroup of K_1 . Thus $K_1/\text{Ker } f_H$ is a finite p -group. On the other hand, G/H is a finite ℓ -group. Since we have an injection $K_1/\text{Ker } f_H \hookrightarrow G/H$, we have $K_1/\text{Ker } f_H = 1$, in other words, $f_H = 1$. Therefore the composite $K_1 \xrightarrow{f} G \xrightarrow{\cong} \varprojlim_H G/H$ is trivial. Hence we have $f = 1$, as desired. \blacksquare

Proof of Theorem 2.1. Since K_1 is an open subgroup of K , we may replace K by K_1 . Take a K_1 -stable \mathbb{Z}_ℓ -lattice Λ of V . Then, $\Lambda/\ell\Lambda$ is a finite abelian group. Therefore, by Lemma 2.3, there exists an open subgroup U of K_1 which acts trivially on $\Lambda/\ell\Lambda$. In other words, the homomorphism $\pi: K_1 \rightarrow GL(\Lambda) \subset GL(V)$ maps U into the subgroup $1 + \ell \text{End}(\Lambda)$. Since U is a closed subgroup of $1 + pM_n(\mathbb{Z}_p)$ and $1 + \ell \text{End}(\Lambda)$ is a pro- ℓ group, by Lemma 2.5, the homomorphism $\pi|_U: U \rightarrow 1 + \ell \text{End}(\Lambda)$ is trivial. Namely, $\pi|_U$ is a trivial representation. \blacksquare

Lemma 2.6 *Let F be a p -adic field and \mathbf{G} a linear algebraic group over F . Then every compact subgroup K of $\mathbf{G}(F)$ can be realized as a closed subgroup of $GL_n(\mathbb{Z}_p)$ for some n .*

Proof. Take an embedding $\mathbf{G} \hookrightarrow GL_m$ defined over F . Since $\mathbf{G}(F)$ is a closed subgroup of $GL_m(F)$, K is also a closed subgroup of $GL_m(F)$. Therefore we have a faithful continuous action of K on F^m . By taking a \mathbb{Q}_p -basis of F , we have a faithful continuous action of K on \mathbb{Q}_p^n for some n . Since K is compact, it is well-known that there is a K -stable \mathbb{Z}_p -lattice in \mathbb{Q}_p^n . Hence we have a continuous injection $K \hookrightarrow GL_n(\mathbb{Z}_p)$. Since K is compact, it is isomorphic to a closed subgroup of $GL_n(\mathbb{Z}_p)$. \blacksquare

Corollary 2.7 *Let F and \mathbf{G} be as in the previous proposition. Let I be a filtered ordered set and $\{K_i\}_{i \in I}$ be a system of compact open subgroups of $\mathbf{G}(F)$ indexed by I .*

Let (π, V) be a (not necessarily finite-dimensional) \mathbb{Q}_ℓ -representation of $\mathbf{G}(F)$ as an abstract group. Assume that there exists an inductive system $\{V_i\}_{i \in I}$ of finite-dimensional \mathbb{Q}_ℓ -vector spaces satisfying the following:

- *For every $i \in I$, V_i is endowed with an action of K_i as an abstract group.*
- *For every $i \in I$, V_i has a K_i -stable \mathbb{Z}_ℓ -lattice.*
- *There exists an isomorphism $\varinjlim_{i \in I} V_i \xrightarrow{\cong} V$ as \mathbb{Q}_ℓ -vector spaces such that the composite $V_i \longrightarrow \varinjlim_{i \in I} V_i \xrightarrow{\cong} V$ is K_i -equivariant for every $i \in I$.*

Then (π, V) is a smooth representation of $\mathbf{G}(F)$.

Proof. Let us take $x \in V$ and show that $\mathrm{Stab}_{\mathbf{G}(F)}(x)$, the stabilizer of x in $\mathbf{G}(F)$, is open. There exists an element $i \in I$ such that x lies in the image of $V_i \longrightarrow V$. Take $y \in V_i$ which is mapped to x . By Theorem 2.1 and Lemma 2.6, V_i is a smooth representation of K_i . Therefore $\mathrm{Stab}_{K_i}(y)$ is open in K_i , hence is open in $\mathbf{G}(F)$. Since $V_i \longrightarrow V$ is K_i -equivariant, we have $\mathrm{Stab}_{K_i}(y) \subset \mathrm{Stab}_{K_i}(x) \subset \mathrm{Stab}_{\mathbf{G}(F)}(x)$. Thus $\mathrm{Stab}_{\mathbf{G}(F)}(x)$ is open in $\mathbf{G}(F)$, as desired. \blacksquare

Remark 2.8 Although we need the corollary above only for the case $F = \mathbb{Q}_p$, we proved it for a general p -adic field F for the completeness.

3 Rapoport-Zink space for $\mathrm{GSp}(4)$

3.1 The Rapoport-Zink space for $\mathrm{GSp}(4)$ and its rigid analytic coverings

In this subsection, we recall basic definitions concerning with Rapoport-Zink spaces. General definitions are given in [RZ96], but here we restrict them to our special case.

Let \mathbb{X} be a 2-dimensional isoclinic p -divisible group over $\overline{\mathbb{F}}_p$ with slope $1/2$, and $\lambda_0: \mathbb{X} \xrightarrow{\cong} \mathbb{X}^\vee$ a (principal) polarization of \mathbb{X} , namely, an isomorphism satisfying $\lambda_0^\vee = -\lambda_0$. Consider the contravariant functor $\mathcal{M}: \mathbf{Nilp} \longrightarrow \mathbf{Set}$ that associates S with the set of isomorphism classes of pairs (X, ρ) consisting of

- a 2-dimensional p -divisible group X over S ,
- and a quasi-isogeny (*cf.* [RZ96, Definition 2.8]) $\rho: \mathbb{X} \otimes_{\overline{\mathbb{F}}_p} \overline{S} \longrightarrow X \otimes_S \overline{S}$,

such that there exists an isomorphism $\lambda: X \longrightarrow X^\vee$ which makes the following

diagram commutative up to multiplication by \mathbb{Q}_p^\times :

$$\begin{array}{ccc} \mathbb{X} \otimes_{\overline{\mathbb{F}}_p} \overline{S} & \xrightarrow{\rho} & X \otimes_S \overline{S} \\ \downarrow \lambda_0 \otimes \text{id} & & \downarrow \lambda \otimes \text{id} \\ \mathbb{X}^\vee \otimes_{\overline{\mathbb{F}}_p} \overline{S} & \xleftarrow{\rho^\vee} & X^\vee \otimes_S \overline{S}. \end{array}$$

Note that such λ is uniquely determined by (X, ρ) up to multiplication by \mathbb{Z}_p^\times and gives a polarization of X . It is proved by Rapoport-Zink that $\check{\mathcal{M}}$ is represented by a special formal scheme (*cf.* [Berk96]) over $\text{Spf } \mathbb{Z}_{p^\infty}$. Moreover, $\check{\mathcal{M}}$ is separated over $\text{Spf } \mathbb{Z}_{p^\infty}$ [Far04, Lemme 2.3.23]. However, $\check{\mathcal{M}}$ is neither quasi-compact nor p -adic. We put $\bar{\mathcal{M}} = \check{\mathcal{M}}_{\text{red}}$, which is a scheme locally of finite type and separated over $\overline{\mathbb{F}}_p$. It is known that $\bar{\mathcal{M}}$ is 1-dimensional (for example, see [Vie08]) and every irreducible component of $\bar{\mathcal{M}}$ is projective over $\overline{\mathbb{F}}_p$ [RZ96, Proposition 2.32]. In particular, $\bar{\mathcal{M}}$ has a locally finite quasi-compact open covering.

Let $D(\mathbb{X})_{\mathbb{Q}} = (N, \Phi)$ be the rational Dieudonné module of \mathbb{X} , which is a 4-dimensional isocrystal over \mathbb{Q}_{p^∞} . The fixed polarization λ_0 gives the alternating pairing $\langle \cdot, \cdot \rangle_{\lambda_0} : N \times N \rightarrow \mathbb{Q}_{p^\infty}(1)$. We define the algebraic group J over \mathbb{Q}_p as follows: for a \mathbb{Q}_p -algebra R , the group $J(R)$ consists of elements $g \in \text{GL}(R \otimes_{\mathbb{Q}_p} N)$ such that

- g commutes with Φ ,
- and g preserves the pairing $\langle \cdot, \cdot \rangle_{\lambda_0}$ up to scalar multiplication, i.e., there exists $c(g) \in R^\times$ such that $\langle gx, gy \rangle_{\lambda_0} = c(g) \langle x, y \rangle_{\lambda_0}$ for every $x, y \in R \otimes_{\mathbb{Q}_p} N$.

It is an inner form of $\text{GSp}(4)$, since $D(\mathbb{X})_{\mathbb{Q}}$ is the isocrystal associated with a basic Frobenius conjugacy class of $\text{GSp}(4)$.

In the sequel, we also denote $J(\mathbb{Q}_p)$ by J . Every element $g \in J$ naturally induces a quasi-isogeny $g : \mathbb{X} \rightarrow \mathbb{X}$ and the following diagram is commutative up to \mathbb{Q}_p^\times -multiplication:

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{g} & \mathbb{X} \\ \downarrow \lambda_0 & & \downarrow \lambda_0 \\ \mathbb{X}^\vee & \xleftarrow{g^\vee} & \mathbb{X}^\vee. \end{array}$$

Therefore, we can define the left action of J on $\check{\mathcal{M}}$ by $g : \check{\mathcal{M}}(S) \rightarrow \check{\mathcal{M}}(S); (X, \rho) \mapsto (X, \rho \circ g^{-1})$.

We denote the Raynaud generic fiber of $\check{\mathcal{M}}$ by $\check{\mathcal{M}}^{\text{rig}}$. It is defined as $t(\check{\mathcal{M}}) \setminus V(p)$, where $t(\check{\mathcal{M}})$ is the adic space associated with $\check{\mathcal{M}}$ (*cf.* [Hub94, Proposition 4.1]). As $\check{\mathcal{M}}$ is separated and special over \mathbb{Z}_{p^∞} , $\check{\mathcal{M}}^{\text{rig}}$ is separated and locally of finite type over $\text{Spa}(\mathbb{Q}_{p^\infty}, \mathbb{Z}_{p^\infty})$. Since $\check{\mathcal{M}}$ has a locally finite quasi-compact open covering, $\check{\mathcal{M}}^{\text{rig}}$ is taut by [Mie10b, Lemma 4.14]. Moreover, by using the period morphism [RZ96, Chapter 5], we can see that $\check{\mathcal{M}}^{\text{rig}}$ is 3-dimensional and smooth over $\text{Spa}(\mathbb{Q}_{p^\infty}, \mathbb{Z}_{p^\infty})$ (*cf.* [RZ96, Proposition 5.17]).

Next we will consider level structures. Let \tilde{X} be the universal p -divisible group over $\check{\mathcal{M}}$ and \tilde{X}^{rig} be the associated p -divisible group over $\check{\mathcal{M}}^{\mathrm{rig}}$. Note that \tilde{X}^{rig} is an étale p -divisible group. Let us fix a polarization $\tilde{\lambda}: \tilde{X} \rightarrow \tilde{X}^\vee$ which is compatible with λ_0 , i.e., satisfies the condition in the definition of $\check{\mathcal{M}}$. Let S be a connected rigid space over \mathbb{Q}_p^∞ (i.e., a connected adic space locally of finite type over $\mathrm{Spa}(\mathbb{Q}_p^\infty, \mathbb{Z}_p^\infty)$), $S \rightarrow \check{\mathcal{M}}^{\mathrm{rig}}$ a morphism over \mathbb{Q}_p^∞ and $\tilde{X}_S^{\mathrm{rig}}$ the pull-back of \tilde{X}^{rig} . Fix a geometric point \bar{x} of S and an isomorphism $T_p(\mu_{p^\infty, S})_{\bar{x}} = \mathbb{Z}_p(1) \cong \mathbb{Z}_p$. Then $\tilde{\lambda}$ induces an alternating bilinear form $\psi_{\tilde{\lambda}}$ on the $\pi_1(S, \bar{x})$ -module $(T_p \tilde{X}_S^{\mathrm{rig}})_{\bar{x}}$:

$$\psi_{\tilde{\lambda}}: (T_p \tilde{X}_S^{\mathrm{rig}})_{\bar{x}} \times (T_p \tilde{X}_S^{\mathrm{rig}})_{\bar{x}} \rightarrow T_p(\mu_{p^\infty, S})_{\bar{x}} \cong \mathbb{Z}_p.$$

Fix a free \mathbb{Z}_p -module L of rank 4 and a perfect alternating bilinear form $\psi_0: L \times L \rightarrow \mathbb{Z}_p$. Put $K_0 = \mathrm{GSp}(L, \psi_0)$, $V = L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $G = \mathrm{GSp}(V, \psi_0)$. Let $T(S, \bar{x})$ be the set consisting of isomorphisms $\eta: L \xrightarrow{\cong} (T_p \tilde{X}_S^{\mathrm{rig}})_{\bar{x}}$ which map ψ_0 to \mathbb{Z}_p^\times -multiples of $\psi_{\tilde{\lambda}}$. It is independent of the choice of $\tilde{\lambda}$ and $T_p(\mu_{p^\infty, S})_{\bar{x}} \cong \mathbb{Z}_p$, since they are unique up to \mathbb{Z}_p^\times -multiplication. Obviously, the groups K_0 and $\pi_1(S, \bar{x})$ naturally act on $T(S, \bar{x})$.

For an open subgroup K of K_0 , a K -level structure of $\tilde{X}_S^{\mathrm{rig}}$ means an element of $(T(S, \bar{x})/K)^{\pi_1(S, \bar{x})}$. Note that, if we change a geometric point \bar{x} to \bar{x}' , the sets $(T(S, \bar{x})/K)^{\pi_1(S, \bar{x})}$ and $(T(S, \bar{x}')/K)^{\pi_1(S, \bar{x}')}$ are naturally isomorphic. Thus the notion of K -level structures is independent of the choice of \bar{x} . The functor that associates S with the set of K -level structures of $\tilde{X}_S^{\mathrm{rig}}$ is represented by a finite Galois étale covering $\check{\mathcal{M}}_K^{\mathrm{rig}} \rightarrow \check{\mathcal{M}}^{\mathrm{rig}}$, whose Galois group is K_0/K . Since $T(S, \bar{x})$ is a K_0 -torsor, $\check{\mathcal{M}}_{K_0}^{\mathrm{rig}}$ coincides with $\check{\mathcal{M}}^{\mathrm{rig}}$. If K' is an open subgroup of K , we have a natural morphism $p_{KK'}: \check{\mathcal{M}}_{K'}^{\mathrm{rig}} \rightarrow \check{\mathcal{M}}_K^{\mathrm{rig}}$. Therefore, we get the projective system of rigid spaces $\{\check{\mathcal{M}}_K^{\mathrm{rig}}\}_K$ indexed by the filtered ordered set of open subgroups of K_0 , which is called the *Rapoport-Zink tower*. Obviously, the group J acts on the projective system $\{\check{\mathcal{M}}_K^{\mathrm{rig}}\}_K$.

Let g be an element of G and K an open subgroup of K_0 which is enough small so that $g^{-1}Kg \subset K_0$. Then we have a natural morphism $\check{\mathcal{M}}_K^{\mathrm{rig}} \rightarrow \check{\mathcal{M}}_{g^{-1}Kg}^{\mathrm{rig}}$ over \mathbb{Q}_p^∞ . If $g \in K_0$, then it is given by $\eta \mapsto \eta \circ g$; for other g , it is more complicated [RZ96, 5.34]. In any case, we get a right action of G on the pro-object “ \varprojlim ” $\check{\mathcal{M}}_K^{\mathrm{rig}}$.

Definition 3.1 We put $H_{\mathrm{RZ}}^i = \varinjlim_K H_c^i(\check{\mathcal{M}}_K^{\mathrm{rig}} \otimes_{\mathbb{Q}_p^\infty} \overline{\mathbb{Q}_p^\infty}, \mathbb{Q}_\ell)$.

Here $H_c^i(\check{\mathcal{M}}_K^{\mathrm{rig}} \otimes_{\mathbb{Q}_p^\infty} \overline{\mathbb{Q}_p^\infty}, \mathbb{Q}_\ell)$ is the compactly supported ℓ -adic cohomology of $\check{\mathcal{M}}_K^{\mathrm{rig}} \otimes_{\mathbb{Q}_p^\infty} \overline{\mathbb{Q}_p^\infty}$ defined in [Hub98]; note that $\check{\mathcal{M}}_K^{\mathrm{rig}}$ is separated and taut. By the constructions above, $G \times J$ acts on H_{RZ}^i on the left (the action of $j \in J$ is given by $(j^{-1})^*$). Obviously the action of G on H_{RZ}^i is smooth. On the other hand, it is known that the action of J on H_{RZ}^i is also smooth. This is due to Berkovich (see [Far04, Corollaire 4.4.7]); see also Remark 5.12, where we give another proof of the smoothness. Hence we get the smooth representation H_{RZ}^i of $G \times J$.

Our main theorem is the following:

Theorem 3.2 (Non-cuspidality) *The smooth representation $H_{\mathrm{RZ}, \overline{\mathbb{Q}_\ell}}^i$ of G has no quasi-cuspidal subquotient unless $i = 2, 3, 4$.*

For the definition of quasi-cuspidal representations, see [Bern84, 1.20].

Theorem 3.2 is proved in Section 5.

3.2 An integral model $\check{\mathcal{M}}_m$ of $\check{\mathcal{M}}_{K_m}^{\mathrm{rig}}$

For an integer $m \geq 1$, let K_m be the kernel of $\mathrm{GSp}(L, \psi_0) \rightarrow \mathrm{GSp}(L/p^m L, \psi_0)$. It is an open subgroup of K_0 . We can describe the definition of K_m -level structures more concretely. As in the previous subsection, we fix a polarization $\tilde{\lambda}$ of \tilde{X}^{rig} which is compatible with λ_0 . It induces the alternating bilinear morphism between finite étale group schemes $\psi_{\tilde{\lambda}}: \tilde{X}^{\mathrm{rig}}[p^m] \times \tilde{X}^{\mathrm{rig}}[p^m] \rightarrow \mu_{p^m}$. Let $\mathcal{S} \rightarrow \check{\mathcal{M}}^{\mathrm{rig}}$ be as in the previous subsection. Then a K_m -level structure of $\tilde{X}_{\mathcal{S}}^{\mathrm{rig}}$ naturally corresponds bijectively to an isomorphism $\eta: L/p^m L \xrightarrow{\cong} \tilde{X}_{\mathcal{S}}^{\mathrm{rig}}[p^m]$ between finite étale group schemes such that there exists an isomorphism $\mathbb{Z}/p^m \mathbb{Z} \xrightarrow{\cong} \mu_{p^m, \mathcal{S}}$ which makes the following diagram commutative:

$$\begin{array}{ccc} L/p^m L \times L/p^m L & \xrightarrow{\psi_0} & \mathbb{Z}/p^m \mathbb{Z} \\ \eta \times \eta \downarrow \cong & & \downarrow \cong \\ \tilde{X}_{\mathcal{S}}^{\mathrm{rig}}[p^m] \times \tilde{X}_{\mathcal{S}}^{\mathrm{rig}}[p^m] & \xrightarrow{\psi_{\tilde{\lambda}}} & \mu_{p^m, \mathcal{S}}. \end{array}$$

For simplicity, we write $\check{\mathcal{M}}_m^{\mathrm{rig}}$ for $\check{\mathcal{M}}_{K_m}^{\mathrm{rig}}$ and p_{mn} for $p_{K_m K_n}$. In this subsection, we construct a formal model $\check{\mathcal{M}}_m$ of $\check{\mathcal{M}}_m^{\mathrm{rig}}$ by following [Man05, §6]. Let \mathcal{S} be a formal scheme of finite type over $\check{\mathcal{M}}^{\mathrm{rig}}$ and denote by $\tilde{X}_{\mathcal{S}}$ the pull-back of \tilde{X} to \mathcal{S} . A Drinfeld m -level structure of $\tilde{X}_{\mathcal{S}}$ is a morphism $\eta: L/p^m L \rightarrow \tilde{X}_{\mathcal{S}}[p^m]$ satisfying the following conditions:

- the image of η gives a full set of sections of $\tilde{X}_{\mathcal{S}}[p^m]$,
- and there exists a morphism $\mathbb{Z}/p^m \mathbb{Z} \rightarrow \mu_{p^m, \mathcal{S}}$ which makes the following diagram commutative:

$$\begin{array}{ccc} L/p^m L \times L/p^m L & \xrightarrow{\psi_0} & \mathbb{Z}/p^m \mathbb{Z} \\ \eta \times \eta \downarrow & & \downarrow \\ \tilde{X}_{\mathcal{S}}[p^m] \times \tilde{X}_{\mathcal{S}}[p^m] & \xrightarrow{\psi_{\tilde{\lambda}}} & \mu_{p^m, \mathcal{S}}. \end{array}$$

It is known that the functor that associates \mathcal{S} with the set of Drinfeld m -level structures of $\tilde{X}_{\mathcal{S}}$ is represented by the formal scheme $\check{\mathcal{M}}_m$ which is finite over $\check{\mathcal{M}}$ (cf. [Man05, Proposition 15]). Note that, unlike the case of Lubin-Tate tower, $\check{\mathcal{M}}_m$ is not necessarily flat over $\check{\mathcal{M}}$. It is easy to show that $\check{\mathcal{M}}_m$ gives a formal model of

$\check{\mathcal{M}}_m^{\mathrm{rig}}$, namely, the Raynaud generic fiber of $\check{\mathcal{M}}_m$ coincides with $\check{\mathcal{M}}_m^{\mathrm{rig}}$. We denote $(\check{\mathcal{M}}_m)_{\mathrm{red}}$ by $\bar{\mathcal{M}}_m$, which is a 1-dimensional scheme over $\bar{\mathbb{F}}_p$.

There is a natural left action of J on $\check{\mathcal{M}}_m$ which is compatible with that on $\check{\mathcal{M}}_m^{\mathrm{rig}}$. On the other hand, the natural action K_0 on $L/p^m L$ induces a right action of K_0 on $\check{\mathcal{M}}_m$, which is compatible with that on $\check{\mathcal{M}}_{K_m}^{\mathrm{rig}}$.

We can also describe $\check{\mathcal{M}}_m$ as a functor from **Nilp** to **Set**; for an object S of **Nilp**, the set $\check{\mathcal{M}}_m(S)$ consists of isomorphism classes of triples (X, ρ, η) , where $(X, \rho) \in \check{\mathcal{M}}_m(S)$ and $\eta: L/p^m L \rightarrow X[p^m]$ is a Drinfeld m -level structure of X . By this description, the action of $j \in J$ on $\check{\mathcal{M}}_m$ is given by $(X, \rho, \eta) \mapsto (X, \rho \circ j^{-1}, \eta)$. On the other hand, the action of $g \in K_0$ on $\check{\mathcal{M}}_m$ is given by $(X, \rho, \eta) \mapsto (X, \rho, \eta \circ g)$.

By [Man04, Lemma 7.2], $\{\check{\mathcal{M}}_m\}_{m \geq 0}$ forms a projective system of formal schemes equipped with the commuting action of J and K_0 .

3.3 Compactly supported cohomology of $\bar{\mathcal{M}}_m$

For $m \geq 0$, we denote the set of quasi-compact open subsets of $\bar{\mathcal{M}}_m$ by \mathcal{Q}_m . It has a natural filtered order by inclusion.

Definition 3.3 For an object \mathcal{F} of $D^b(\bar{\mathcal{M}}_m, \mathbb{Z}_\ell)$ or $D^b(\bar{\mathcal{M}}_m, \mathbb{Q}_\ell)$, we put

$$H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}) = \varinjlim_{U \in \mathcal{Q}_m} H_c^i(U, \mathcal{F}|_U).$$

Assume that \mathcal{F} has a J -equivariant structure, namely, for every $g \in J$ an isomorphism $\varphi_g: g^* \mathcal{F} \xrightarrow{\cong} \mathcal{F}$ is given such that $\varphi_{gg'} = \varphi_{g'} \circ g'^* \varphi_g$ for every $g, g' \in J$. Then J naturally acts on $H_c^i(\bar{\mathcal{M}}_m, \mathcal{F})$ on the right. Therefore we get a left action of J on $H_c^i(\bar{\mathcal{M}}_m, \mathcal{F})$ by taking the inverse $J \rightarrow J; g \mapsto g^{-1}$.

Theorem 3.4 Let \mathcal{F}° be an object of $D_c^b(\bar{\mathcal{M}}_m, \mathbb{Z}_\ell)$ and \mathcal{F} the object of $D_c^b(\bar{\mathcal{M}}_m, \mathbb{Q}_\ell)$ associated with \mathcal{F}° . Assume that we are given a J -equivariant structure of \mathcal{F}° (thus \mathcal{F} also has a J -equivariant structure). Then $H_c^i(\bar{\mathcal{M}}_m, \mathcal{F})$ is a finitely generated smooth J -representation.

Proof. Let U be an element of \mathcal{Q}_m . By [Far04, Proposition 2.3.11], there exists a compact open subgroup K_U of J which stabilizes U . Then $H_c^i(U, \mathcal{F}|_U)$ is a finite-dimensional \mathbb{Q}_ℓ -vector space endowed with the action of K_U and has the K_U -stable \mathbb{Z}_ℓ -lattice $\mathrm{Im}(H_c^i(U, \mathcal{F}^\circ|_U) \rightarrow H_c^i(U, \mathcal{F}|_U))$. Therefore $H_c^i(\bar{\mathcal{M}}_m, \mathcal{F})$ is a smooth J -representation by Corollary 2.7.

To prove that $H_c^i(\bar{\mathcal{M}}_m, \mathcal{F})$ is finitely generated, we may assume $m = 0$, for $H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}) = H_c^i(\bar{\mathcal{M}}_0, p_{0m*} \mathcal{F})$. In this case, we can use the similar method as in [Far04, Proposition 4.4.13]. Let us explain the argument briefly. By [Far04, Théorème 2.4.13], there exists $W \in \mathcal{Q}_0$ such that $\bigcup_{g \in J} gW = \bar{\mathcal{M}}_0$. We put $K = \{g \in J \mid gW = W\}$ and $\Omega = \{g \in J \mid gW \cap W \neq \emptyset\}$. As in the proof of [Far04, Proposition 4.4.13], K is a compact open subgroup of J and Ω is a compact

subset of J . For $\alpha = ([g_1], \dots, [g_n]) \in (J/K)^n$, we put $W_\alpha = g_1 W \cap \dots \cap g_n W$ and $K_\alpha = \bigcap_{j=1}^n g_j K g_j^{-1}$. For an open covering $\{gW\}_{g \in J/K}$, we can associate the Čech spectral sequence

$$E_1^{r,s} = \bigoplus_{\alpha \in (J/K)^{-r+1}} H_c^s(W_\alpha, \mathcal{F}|_{W_\alpha}) \implies H_c^{r+s}(\bar{\mathcal{M}}_0, \mathcal{F}).$$

Consider the diagonal action of J on $(J/K)^{-r+1}$. The coset

$$J \setminus \{\alpha \in (J/K)^{-r+1} \mid W_\alpha \neq \emptyset\}$$

is finite; indeed, if $W_\alpha \neq \emptyset$ for $\alpha = ([g_1], \dots, [g_{-r+1}]) \in (J/K)^{-r+1}$, then $g_1^{-1}\alpha \in \{1\} \times \Omega/K \times \dots \times \Omega/K$, which is a finite set.

Take a system of representatives $\alpha_1, \dots, \alpha_n$ of the coset above. Then there is a natural isomorphism $\bigoplus_{\alpha \in J\alpha_j} H_c^s(W_\alpha, \mathcal{F}|_{W_\alpha}) \cong \text{c-Ind}_{K_{\alpha_j}}^J H_c^s(W_{\alpha_j}, \mathcal{F}|_{W_{\alpha_j}})$. Hence $E_1^{r,s} \cong \bigoplus_{j=1}^n \text{c-Ind}_{K_{\alpha_j}}^J H_c^s(W_{\alpha_j}, \mathcal{F}|_{W_{\alpha_j}})$ is a finitely generated J -module, since the cohomology $H_c^s(W_{\alpha_j}, \mathcal{F}|_{W_{\alpha_j}})$ is finite-dimensional for each j . By this and the fact that a finitely generated smooth J -module is noetherian [Bern84, Remarque 3.12], we conclude that $H_c^i(\bar{\mathcal{M}}_0, \mathcal{F})$ is finitely generated. \blacksquare

Lemma 3.5 *Let \mathcal{F} be an object of $D_c^b(\bar{\mathcal{M}}_m, \mathbb{Q}_\ell)$ with a K_0/K_m -equivariant structure. Let n be an integer with $0 \leq n \leq m$ and put $\mathcal{G} = (p_{nm*}\mathcal{F})^{K_n/K_m}$. Then we have $H_c^i(\bar{\mathcal{M}}_m, \mathcal{F})^{K_n/K_m} = H_c^i(\bar{\mathcal{M}}_n, \mathcal{G})$.*

Proof. Since the cardinality of K_n/K_m is prime to ℓ , $(-)^{K_n/K_m}$ commutes with H_c^i . Therefore, we have

$$\begin{aligned} H_c^i(\bar{\mathcal{M}}_m, \mathcal{F})^{K_n/K_m} &= \varinjlim_{U \in \mathcal{Q}_m} H_c^i(U, \mathcal{F}|_U)^{K_n/K_m} = \varinjlim_{V \in \mathcal{Q}_n} H_c^i(p_{nm}^{-1}(V), \mathcal{F}|_{p_{nm}^{-1}(V)})^{K_n/K_m} \\ &= \varinjlim_{V \in \mathcal{Q}_n} H_c^i(V, p_{nm*}(\mathcal{F}|_{p_{nm}^{-1}(V)}))^{K_n/K_m} = \varinjlim_{V \in \mathcal{Q}_n} H_c^i(V, (p_{nm*}(\mathcal{F}|_{p_{nm}^{-1}(V)}))^{K_n/K_m}) \\ &= \varinjlim_{V \in \mathcal{Q}_n} H_c^i(V, \mathcal{G}|_V) = H_c^i(\bar{\mathcal{M}}_n, \mathcal{G}). \end{aligned} \quad \blacksquare$$

Definition 3.6 *A system of coefficients over the tower $\{\bar{\mathcal{M}}_m\}_{m \geq 0}$ is the data $\mathcal{F} = \{\mathcal{F}_m\}_{m \geq 0}$ where \mathcal{F}_m is an object of $D_c^b(\bar{\mathcal{M}}_m, \mathbb{Q}_\ell)$ with a K_0/K_m -equivariant structure such that $(p_{nm*}\mathcal{F})^{K_n/K_m} = \mathcal{F}_n$ for every integers m, n with $0 \leq n \leq m$. Then, by Lemma 3.5, we have $H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_m)^{K_n/K_m} = H_c^i(\bar{\mathcal{M}}_n, \mathcal{F}_n)$. We put $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}) = \varinjlim_m H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_m)$.*

If each \mathcal{F}_m is endowed with a J -equivariant structure which commutes with the given K_0/K_m -equivariant structure, and for every $0 \leq n \leq m$ the J -equivariant structures on \mathcal{F}_m and \mathcal{F}_n are compatible under the identification $(p_{nm*}\mathcal{F}_m)^{K_n/K_m} = \mathcal{F}_n$, then we say that we have a J -equivariant structure on \mathcal{F} . Such a structure naturally induces the action of J on $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F})$.

By replacing “ $D_c^b(\bar{\mathcal{M}}_m, \mathbb{Q}_\ell)$ ” with “ $D_c^b(\bar{\mathcal{M}}_m, \mathbb{Z}_\ell)$ ”, we may also define a *system of integral coefficients* \mathcal{F}° over $\{\bar{\mathcal{M}}_m\}_{m \geq 0}$, the cohomology $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^\circ)$ and a J -equivariant structure on \mathcal{F}° .

Corollary 3.7 *Let \mathcal{F}° be a system of integral coefficients over $\{\bar{\mathcal{M}}_m\}_{m \geq 0}$ with a J -equivariant structure and \mathcal{F} the system of coefficients associated with \mathcal{F}° . Then $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F})$ is a smooth $K_0 \times J$ -representation and $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F})^{K_m}$ is a finitely generated smooth J -representation for every integer $m \geq 0$.*

Proof. The smoothness is clear from Theorem 3.4 and the definition of $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F})$. Since $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F})^{K_m} = H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_m)$, the second assertion also follows from Theorem 3.4. \blacksquare

4 Shimura variety and p -adic uniformization

In this section, we introduce certain Shimura varieties (Siegel threefolds) related to our Rapoport-Zink tower. Let us fix a 4-dimensional \mathbb{Q} -vector space V' and an alternating perfect pairing $\psi': V' \times V' \rightarrow \mathbb{Q}$. For an integer $m \geq 0$ and a compact open subgroup $K^p \subset \mathrm{GSp}(V'_{\mathbb{A}^\infty, p}) = \mathrm{GSp}(V'_{\mathbb{A}^\infty, p}, \psi'_{\mathbb{A}^\infty, p})$, consider the functor Sh_{m, K^p} from the category of locally noetherian \mathbb{Z}_p^∞ -schemes to the category of sets that associates S with the set of isomorphism classes of quadruples $(A, \lambda, \eta^p, \eta_p)$ where

- A is a projective abelian surface over S up to prime-to- p isogeny,
- $\lambda: A \rightarrow A^\vee$ is a prime-to- p polarization,
- η^p is a K^p -level structure of A ,
- and $\eta_p: L/p^m L \rightarrow A[p^m]$ is a Drinfeld m -level structure

(for the detail, see [Kot92, §5]). Two quadruples $(A, \lambda, \eta^p, \eta_p)$ and $(A', \lambda', \eta'^p, \eta'_p)$ are said to be isomorphic if there exists a prime-to- p isogeny from A to A' which carries λ to a $\mathbb{Z}_{(p)}^\times$ -multiple of λ' , η^p to η'^p and η_p to η'_p . We put $\mathrm{Sh}_{K^p} = \mathrm{Sh}_{0, K^p}$. It is known that if K^p is sufficiently small, Sh_{m, K^p} is represented by a quasi-projective scheme over \mathbb{Z}_p^∞ with smooth generic fiber. In the sequel, we always assume that K^p is enough small so that Sh_{m, K^p} is representable. We denote the special fiber of Sh_{m, K^p} (resp. Sh_{K^p}) by $\overline{\mathrm{Sh}}_{m, K^p}$ (resp. $\overline{\mathrm{Sh}}_{K^p}$).

For a compact open subgroup K'^p contained in K^p and an integer $m' \geq m$, we have the natural morphism $\mathrm{Sh}_{m', K'^p} \rightarrow \mathrm{Sh}_{m, K^p}$. This is a finite morphism and is moreover étale if $m' = m$.

Next we recall the p -adic uniformization theorem, which gives a relation between $\bar{\mathcal{M}}$ and Sh_{K^p} . Let us fix a polarized abelian surface (A_0, λ_{A_0}) over $\overline{\mathbb{F}}_p$ such that $A_0[p^\infty]$ is an isoclinic p -divisible group with slope $1/2$. Note that such (A_0, λ_{A_0}) exists; for example, we can take $(A_0, \lambda_{A_0}) = (E^2, \lambda_E^2)$, where E is a supersingular elliptic curve over $\overline{\mathbb{F}}_p$ and λ_E is a polarization of E . By definition, the rational Dieudonné module $D(A_0[p^\infty])_{\mathbb{Q}}$ is isomorphic to $D(\mathbb{X})_{\mathbb{Q}}$. Thus, by the subsequent lemma, there is an isomorphism of isocrystals $D(A_0[p^\infty])_{\mathbb{Q}} \cong D(\mathbb{X})_{\mathbb{Q}}$ which preserves the natural polarizations.

Lemma 4.1 *We use the notation in [RR96, §1]. Let $d \geq 1$ be an integer.*

- i) *Let b be an element of $B(\mathrm{GSp}_{2d})$ and b' the image of b under the natural map $B(\mathrm{GSp}_{2d}) \rightarrow B(\mathrm{GL}_{2d})$. Then b is basic if and only if b' is basic.*

ii) The map $B(\mathrm{GSp}_{2d})_{\mathrm{basic}} \longrightarrow B(\mathrm{GL}_{2d})_{\mathrm{basic}}$ induced from i) is an injection.

Proof. Note that the center of GSp_{2d} coincides with that of GL_{2d} . Thus i) is clear, since b (resp. b') is basic if and only if the slope morphism $\nu_b: \mathbb{D} \longrightarrow \mathrm{GSp}_{2d}$ (resp. $\nu_{b'}: \mathbb{D} \xrightarrow{\nu_b} \mathrm{GSp}_{2d} \hookrightarrow \mathrm{GL}_{2d}$) factors through the center of GSp_{2d} (resp. GL_{2d}).

We prove ii). By [RR96, Theorem 1.15], it suffices to show that the natural map $\pi_1(\mathrm{GSp}_{2d}) \longrightarrow \pi_1(\mathrm{GL}_{2d})$ is injective. Take a maximal torus T (resp. T') of GSp_{2d} (resp. GL_{2d}) such that $T \subset T'$. Then, since Sp_{2d} (resp. SL_{2d}) is simply connected, $\pi_1(\mathrm{GSp}_{2d})$ (resp. $\pi_1(\mathrm{GL}_{2d})$) can be identified with the quotient of $X_*(T)$ (resp. $X_*(T')$) induced by $c: T \twoheadrightarrow \mathbb{G}_m$ (resp. $\det: T' \twoheadrightarrow \mathbb{G}_m$), where c denotes the similitude character of GSp_{2d} . In particular, both $\pi_1(\mathrm{GSp}_{2d})$ and $\pi_1(\mathrm{GL}_{2d})$ are isomorphic to \mathbb{Z} .

The commutative diagram

$$\begin{array}{ccc} \mathrm{GSp}_{2d} & \xrightarrow{c} & \mathbb{G}_m \\ \downarrow & & \downarrow z \mapsto z^d \\ \mathrm{GL}_{2d} & \xrightarrow{\det} & \mathbb{G}_m \end{array}$$

induces the commutative diagram

$$\begin{array}{ccc} X_*(T) & \twoheadrightarrow & X_*(\mathbb{G}_m) = \pi_1(\mathrm{GSp}_{2d}) \\ \downarrow & & \downarrow \times d \\ X_*(T') & \twoheadrightarrow & X_*(\mathbb{G}_m) = \pi_1(\mathrm{GL}_{2d}). \end{array}$$

In particular, the natural map $\pi_1(\mathrm{GSp}_{2d}) \longrightarrow \pi_1(\mathrm{GL}_{2d})$ is injective. ■

Therefore, there is a quasi-isogeny $\mathbb{X} \longrightarrow A[p^\infty]$ preserving polarizations. If we replace (\mathbb{X}, λ_0) by the polarized p -divisible group $(A_0[p^\infty], \lambda_{A_0})$ associated with (A_0, λ_{A_0}) , the G -representation H_{RZ}^i remains unchanged. Thus, in order to prove Theorem 3.2, we may assume that $(\mathbb{X}, \lambda_0) = (A_0[p^\infty], \lambda_{A_0})$. In the remaining part of this article, we always assume it. Moreover, we fix an isomorphism $H_1(A_0, \mathbb{A}^{\infty, p}) \cong V'_{\mathbb{A}^{\infty, p}}$ preserving alternating pairings.

Denote the isogeny class of (A_0, λ_{A_0}) by ϕ and put $I^\phi = \mathrm{Aut}(\phi)$. We have natural group homomorphisms $I^\phi \hookrightarrow J$ and $I^\phi \hookrightarrow \mathrm{Aut}(H_1(A_0, \mathbb{A}^{\infty, p})) = \mathrm{GSp}(V'_{\mathbb{A}^{\infty, p}})$. These are injective.

Let Y_{K^p} be the reduced closed subscheme of $\overline{\mathrm{Sh}}_{K^p}$ such that $Y_{K^p}(\overline{\mathbb{F}}_p)$ consists of triples (A, λ, η^p) where the p -divisible group associated with (A, λ) is isogenous to (\mathbb{X}, λ_0) . It is the basic (or supersingular) stratum in the Newton stratification of $\overline{\mathrm{Sh}}_{K^p}$. Note that $(A, \lambda, \eta^p) \in \overline{\mathrm{Sh}}_{K^p}(\overline{\mathbb{F}}_p)$ belongs to $Y_{K^p}(\overline{\mathbb{F}}_p)$ if and only if $(A, \lambda) \in \phi$ ([Far04, Proposition 3.1.8], [Kot92, §7]). We denote the formal completion of Sh_{K^p} along Y_{K^p} by $(\mathrm{Sh}_{K^p})_{/Y_{K^p}}^\wedge$.

Now we can state the p -adic uniformization theorem:

Theorem 4.2 ([RZ96, Theorem 6.30]) *There exists a natural isomorphism of formal schemes:*

$$\theta_{K^p}: I^\phi \backslash (\check{\mathcal{M}} \times \mathrm{GSp}(V'_{\mathbb{A}^\infty, p})/K^p) \xrightarrow{\cong} (\mathrm{Sh}_{K^p})^\wedge_{Y_{K^p}}.$$

In the left hand side, I^ϕ acts on $\check{\mathcal{M}}$ through $I^\phi \hookrightarrow J$ and acts on $\mathrm{GSp}(V'_{\mathbb{A}^\infty, p})/K^p$ through $I^\phi \hookrightarrow \mathrm{GSp}(V'_{\mathbb{A}^\infty, p})$.

The isomorphisms $\{\theta_{K^p}\}_{K^p}$ are compatible with change of K^p . (It is also compatible with the Hecke action of $\mathrm{GSp}_4(V'_{\mathbb{A}^\infty, p})$, but we do not use it.)

Let us briefly recall the construction of the isomorphism θ_{K^p} . Take a lift $(\tilde{\mathbb{X}}, \tilde{\lambda}_0)$ of (\mathbb{X}, λ_0) over \mathbb{Z}_{p^∞} (such a lift is unique up to isomorphism). Then, by the Serre-Tate theorem, the lift $(\tilde{A}_0, \tilde{\lambda}_{A_0})$ of (A_0, λ_{A_0}) is canonically determined. Let S be an object of \mathbf{Nilp} , $(X, \rho) \in \check{\mathcal{M}}(S)$ and $[g] \in \mathrm{GSp}(V'_{\mathbb{A}^\infty, p})/K^p$. Then ρ extends uniquely to the quasi-isogeny $\tilde{\rho}: \tilde{\mathbb{X}} \times_{\mathbb{Z}_{p^\infty}} S \rightarrow X$. We can see that there exist a polarized abelian variety (A, λ) and a p -quasi-isogeny $\tilde{A}_0 \times_{\mathbb{Z}_{p^\infty}} S \rightarrow A$ preserving polarizations, such that the associated quasi-isogeny $\tilde{A}_0[p^\infty] \times_{\mathbb{Z}_{p^\infty}} S \rightarrow A[p^\infty]$ coincides with $\tilde{\rho}$. The fixed isomorphism $H_1(A_0, \mathbb{A}^{\infty, p}) \cong V'_{\mathbb{A}^\infty, p}$ naturally induces a K^p -level structure η of A . The morphism θ_{K^p} is given by $\theta_{K^p}((X, \rho), [g]) = (A, \lambda, \eta \circ g)$.

By composing the morphism $\check{\mathcal{M}} \rightarrow \check{\mathcal{M}} \times \mathrm{GSp}(V'_{\mathbb{A}^\infty, p})/K^p$; $x \mapsto (x, [\mathrm{id}])$, we get a morphism $\check{\mathcal{M}} \rightarrow (\mathrm{Sh}_{K^p})^\wedge_{Y_{K^p}}$, which is also denoted by θ_{K^p} . For $U \in \mathcal{Q}_0$, we denote the image of U under θ_{K^p} by $Y_{K^p}(U)$. It is an open subset of Y_{K^p} .

Proposition 4.3 *Let U be an element of \mathcal{Q}_0 . Then for a sufficiently small compact open subgroup K^p of $\mathrm{GSp}(V'_{\mathbb{A}^\infty, p})$, θ_{K^p} induces an isomorphism $U \xrightarrow{\cong} Y_{K^p}(U)$. Moreover, if we denote the open formal subscheme of $\check{\mathcal{M}}$ (resp. $(\mathrm{Sh}_{K^p})^\wedge_{Y_{K^p}}$) whose underlying topological space is U (resp. $Y_{K^p}(U)$) by $\check{\mathcal{M}}/U$ (resp. $(\mathrm{Sh}_{K^p})^\wedge_{Y_{K^p}(U)}$), then θ_{K^p} induces an isomorphism $\theta_{K^p}: \check{\mathcal{M}}/U \xrightarrow{\cong} (\mathrm{Sh}_{K^p})^\wedge_{Y_{K^p}(U)}$.*

Proof. The proof is similar to [Far04, Corollaire 3.1.4]. Put $\Gamma_{K^p} = I^\phi \cap K^p$, where the intersection is taken in $\mathrm{GSp}(V'_{\mathbb{A}^\infty, p})$. It is known that Γ_{K^p} is discrete and torsion-free [RZ96]. By Theorem 4.2, θ_{K^p} gives an isomorphism from $\Gamma_{K^p} \backslash \check{\mathcal{M}}$ to an open and closed formal subscheme of $(\mathrm{Sh}_{K^p})^\wedge_{Y_{K^p}}$. By the same method as in [Far04, Lemme 3.1.2, Proposition 3.1.3], we can see that every element $\gamma \in \Gamma_{K^p}$ other than 1 satisfies $\gamma \cdot U \cap U = \emptyset$ if K^p is sufficiently small. For such K^p , the natural morphism $\check{\mathcal{M}}/U \rightarrow \Gamma_{K^p} \backslash \check{\mathcal{M}}$ is an open immersion. Thus we have an open immersion $\check{\mathcal{M}}/U \hookrightarrow \Gamma_{K^p} \backslash \check{\mathcal{M}} \xrightarrow[\cong]{\theta_{K^p}} (\mathrm{Sh}_{K^p})^\wedge_{Y_{K^p}}$, whose image is $(\mathrm{Sh}_{K^p})^\wedge_{Y_{K^p}(U)}$. ■

Next we consider the case with Drinfeld level structures at p . Let Y_{m, K^p} be the closed subscheme of $\overline{\mathrm{Sh}}_{m, K^p}$ obtained as the inverse image of Y_{K^p} under $\overline{\mathrm{Sh}}_{m, K^p} \rightarrow \overline{\mathrm{Sh}}_{K^p}$. By the construction of θ_{K^p} described above, we have the following result:

Corollary 4.4 *Let $m \geq 0$ be an integer. We can construct naturally a morphism $\theta_{m,K^p} : \check{\mathcal{M}}_m \longrightarrow (\mathrm{Sh}_{m,K^p})^\wedge_{Y_{m,K^p}}$ which makes the following diagram cartesian:*

$$\begin{array}{ccc} \check{\mathcal{M}}_m & \xrightarrow{\theta_{m,K^p}} & (\mathrm{Sh}_{m,K^p})^\wedge_{Y_{m,K^p}} \\ \downarrow p_{m0} & & \downarrow \\ \check{\mathcal{M}} & \xrightarrow{\theta_{K^p}} & (\mathrm{Sh}_{K^p})^\wedge_{Y_{K^p}}. \end{array}$$

In particular, the similar result as Proposition 4.3 holds for θ_{m,K^p} ; that is, for $U \in \mathcal{Q}_m$, θ_{m,K^p} induces $(\check{\mathcal{M}}_m)_{/U} \xrightarrow{\cong} (\mathrm{Sh}_{m,K^p})^\wedge_{Y_{m,K^p}(U)}$ if K^p is sufficiently small.

5 Proof of the non-cuspidality result

5.1 The system of coefficients $\mathcal{F}^{[h]}$, $\mathcal{F}^{(h)}$

Definition 5.1 Let $m \geq 1$ and $0 \leq h \leq 2$ be integers. We denote by $\mathcal{S}_{m,h}$ the set of direct summands of $L/p^m L$ of rank $4-h$, and by $\mathcal{S}_{m,h}^{\mathrm{coi}}$ the subset of $\mathcal{S}_{m,h}$ consisting of coisotropic direct summands (recall that $I \in \mathcal{S}_{m,h}$ is said to be coisotropic if $I^\perp \subset I$). Put $\mathcal{S}_m = \bigcup_{h=0}^2 \mathcal{S}_{m,h}$ and $\mathcal{S}_m^{\mathrm{coi}} = \bigcup_{h=0}^2 \mathcal{S}_{m,h}^{\mathrm{coi}}$.

For $I \in \mathcal{S}_{m,h}$, let $\overline{\mathrm{Sh}}_{m,K^p,[I]}$ be the $\overline{\mathbb{F}}_p$ -scheme defined by

$$\overline{\mathrm{Sh}}_{m,K^p,[I]}(S) = \{(A, \lambda, \eta^p, \eta_p) \in \overline{\mathrm{Sh}}_{m,K^p,[I]}(S) \mid I \subset \mathrm{Ker} \eta_p\}.$$

Clearly it is a closed subscheme of $\overline{\mathrm{Sh}}_{m,K^p}$. Similarly, we can define the closed formal subscheme $\check{\mathcal{M}}_{m,[I]}$ of $\check{\mathcal{M}}_m \otimes_{\mathbb{Z}_p^\infty} \overline{\mathbb{F}}_p$. Obviously, $\check{\mathcal{M}}_{m,[I]}$ is stable under the action of J on $\check{\mathcal{M}}_m$.

We denote by $Y_{m,K^p,[I]}$ the closed subscheme of $\overline{\mathrm{Sh}}_{m,K^p,[I]}$ obtained as the inverse image of Y_{m,K^p} . As Corollary 4.4, we have the following cartesian diagram of formal schemes:

$$\begin{array}{ccc} \check{\mathcal{M}}_{m,[I]} & \longrightarrow & (\overline{\mathrm{Sh}}_{m,K^p,[I]})^\wedge_{Y_{m,K^p,[I]}} \\ \downarrow & & \downarrow \\ \check{\mathcal{M}}_m & \xrightarrow{\theta_{m,K^p}} & (\mathrm{Sh}_{m,K^p})^\wedge_{Y_{m,K^p}}. \end{array}$$

Definition 5.2 For $I \in \mathcal{S}_m$, we put

$$\overline{\mathrm{Sh}}_{m,K^p,(I)} = \overline{\mathrm{Sh}}_{m,K^p,[I]} \setminus \bigcup_{I' \in \mathcal{S}_m, I \subsetneq I'} \overline{\mathrm{Sh}}_{m,K^p,[I]},$$

which is an open subscheme of $\overline{\mathrm{Sh}}_{m,K^p,[I]}$, and thus is a subscheme of $\overline{\mathrm{Sh}}_{m,K^p}$. Moreover, for an integer h with $0 \leq h \leq 2$, we put $\overline{\mathrm{Sh}}_{m,K^p}^{[h]} = \bigcup_{I \in \mathcal{S}_{m,h}} \overline{\mathrm{Sh}}_{m,K^p,[I]}$ and

$\overline{\mathrm{Sh}}_{m,K^p}^{(h)} = \bigcup_{I \in \mathcal{S}_{m,h}} \overline{\mathrm{Sh}}_{m,K^p,(I)}$. The former is a closed subscheme of $\overline{\mathrm{Sh}}_{m,K^p}$, which is the scheme theoretic image of $\coprod_{I \in \mathcal{S}_{m,h}} \overline{\mathrm{Sh}}_{m,K^p,[I]} \longrightarrow \overline{\mathrm{Sh}}_{m,K^p}$. The latter is an open subscheme of $\overline{\mathrm{Sh}}_{m,K^p}^{[h]}$, since $\overline{\mathrm{Sh}}_{m,K^p}^{(h)} = \overline{\mathrm{Sh}}_{m,K^p}^{[h]} \setminus \overline{\mathrm{Sh}}_{m,K^p}^{[h-1]}$ (if $h = 0$, we put $\overline{\mathrm{Sh}}_{m,K^p}^{[-1]} = \emptyset$).

Lemma 5.3 i) Let $x = (A, \lambda, \eta^p, \eta_p)$ be an element of $\overline{\mathrm{Sh}}_{m,K^p}(\overline{\mathbb{F}}_p)$. Then, for $I \in \mathcal{S}_m$, $x \in \overline{\mathrm{Sh}}_{m,K^p,(I)}(\overline{\mathbb{F}}_p)$ if and only if $I = \mathrm{Ker} \eta_p$. For an integer h with $0 \leq h \leq 2$, $x \in \overline{\mathrm{Sh}}_{m,K^p}^{[h]}(\overline{\mathbb{F}}_p)$ (resp. $x \in \overline{\mathrm{Sh}}_{m,K^p}^{(h)}(\overline{\mathbb{F}}_p)$) if and only if $\mathrm{rank}_{\mathbb{F}_p} A[p] \leq h$ (resp. $\mathrm{rank}_{\mathbb{F}_p} A[p] = h$).

ii) For every integer h with $0 \leq h \leq 2$, we have $\overline{\mathrm{Sh}}_{m,K^p}^{(h)} = \coprod_{I \in \mathcal{S}_{m,h}} \overline{\mathrm{Sh}}_{m,K^p,(I)}$ as schemes.

iii) We have $(\overline{\mathrm{Sh}}_{m,K^p}^{[2]})_{\mathrm{red}} = (\overline{\mathrm{Sh}}_{m,K^p})_{\mathrm{red}}$ and $(\overline{\mathrm{Sh}}_{m,K^p}^{[0]})_{\mathrm{red}} = (Y_{m,K^p})_{\mathrm{red}}$.

Proof. Let us prove i). Put $X = A[p^\infty]$. Then there is an exact sequence $0 \longrightarrow X_0 \longrightarrow X \longrightarrow X_{\mathrm{\acute{e}t}} \longrightarrow 0$, where X_0 is a connected p -divisible group and $X_{\mathrm{\acute{e}t}}$ is an étale p -divisible group. By [HT01, Lemma II.2.1], $\mathrm{Ker} \eta_p$ is a direct summand of $L/p^m L$ and $(L/p^m L)/\mathrm{Ker} \eta_p \longrightarrow X_{\mathrm{\acute{e}t}}[p^m]$ is an isomorphism. Thus $\mathrm{Ker} \eta_p \in \mathcal{S}_{m,r}$, where $r = \mathrm{rank}_{\mathbb{Z}/p^m \mathbb{Z}} X_{\mathrm{\acute{e}t}}[p^m] = \mathrm{rank}_{\mathbb{F}_p} A[p] \leq 2$. By this, all the claims in i) are immediate.

By i), $\overline{\mathrm{Sh}}_{m,K^p}^{(h)}$ coincides with $\coprod_{I \in \mathcal{S}_{m,h}} \overline{\mathrm{Sh}}_{m,K^p,(I)}$ as a set; thus to prove ii) it suffices to show that $\overline{\mathrm{Sh}}_{m,K^p,(I)}$ is closed (hence open) in $\overline{\mathrm{Sh}}_{m,K^p}^{(h)}$ for every $I \in \mathcal{S}_{m,h}$. It is clear from $\overline{\mathrm{Sh}}_{m,K^p,(I)} = \overline{\mathrm{Sh}}_{m,K^p,[I]} \cap \overline{\mathrm{Sh}}_{m,K^p}^{(h)}$.

The former equality in iii) follows immediately from i). We will prove the latter. For $x = (A, \lambda, \eta^p, \eta_p) \in \overline{\mathrm{Sh}}_{m,K^p}^{[0]}(\overline{\mathbb{F}}_p)$, $X = A[p^\infty]$ has no étale part by i). Since $X^\vee \cong X$, X has no multiplicative part. Therefore X is isoclinic of slope $1/2$; indeed, if a Newton polygon with the terminal point $(4, 2)$ has neither slope 0 part nor slope 1 part, then it is a line of slope $1/2$. Thus, by Lemma 4.1, there is a quasi-isogeny $\mathbb{X} \longrightarrow X$ preserving polarizations; namely, $x \in Y_{m,K^p}(\overline{\mathbb{F}}_p)$. The opposite inclusion is clear. \blacksquare

Remark 5.4 The latter part of iii) in Lemma 5.3 is the only place where the same argument does not work in the case $\mathrm{GSp}(2d)$ with $d \geq 3$.

Definition 5.5 Let $m \geq 1$ be an integer. Fix a compact open subgroup K^p of $\mathrm{GSp}(V'_{\mathbb{A}^\infty,p})$. For $I \in \mathcal{S}_m$, denote the natural immersion $\overline{\mathrm{Sh}}_{m,K^p,(I)} \hookrightarrow \overline{\mathrm{Sh}}_{m,K^p}$ by $j_{m,I}$. For an integer h with $0 \leq h \leq 2$, denote the natural immersions $\overline{\mathrm{Sh}}_{m,K^p}^{[h]} \hookrightarrow \overline{\mathrm{Sh}}_{m,K^p}$ and $\overline{\mathrm{Sh}}_{m,K^p}^{(h)} \hookrightarrow \overline{\mathrm{Sh}}_{m,K^p}$ by $j_m^{[h]}$ and $j_m^{(h)}$, respectively.

We define $\mathcal{F}_{m,I}^\circ$, $\mathcal{F}_{m,I}$, $\mathcal{F}_m^{\circ[h]}$, $\mathcal{F}_m^{[h]}$, $\mathcal{F}_m^{\circ(h)}$ and $\mathcal{F}_m^{(h)}$ as follows:

$$\begin{aligned} \mathcal{F}_{m,I}^\circ &= \theta_m^*(Rj_{m,I*} Rj_{m,I}^! R\psi \mathbb{Z}_\ell)|_{Y_{m,K^p}}, & \mathcal{F}_{m,I} &= \theta_m^*(Rj_{m,I*} Rj_{m,I}^! R\psi \mathbb{Q}_\ell)|_{Y_{m,K^p}}, \\ \mathcal{F}_m^{\circ[h]} &= \theta_m^*(Rj_{m*}^{[h]} Rj_m^{[h]!} R\psi \mathbb{Z}_\ell)|_{Y_{m,K^p}}, & \mathcal{F}_m^{[h]} &= \theta_m^*(Rj_{m*}^{[h]} Rj_m^{[h]!} R\psi \mathbb{Q}_\ell)|_{Y_{m,K^p}}, \\ \mathcal{F}_m^{\circ(h)} &= \theta_m^*(Rj_{m*}^{(h)} Rj_m^{(h)!} R\psi \mathbb{Z}_\ell)|_{Y_{m,K^p}}, & \mathcal{F}_m^{(h)} &= \theta_m^*(Rj_{m*}^{(h)} Rj_m^{(h)!} R\psi \mathbb{Q}_\ell)|_{Y_{m,K^p}}. \end{aligned}$$

Here $\theta_m: \mathcal{M}_m \rightarrow Y_{m,K^p}$ is the morphism induced from θ_{m,K^p} in Corollary 4.4.

These are independent of the choice of K^p ; indeed, for another compact open subgroup K'^p contained in K^p , the natural map $\mathrm{Sh}_{m,K'^p} \rightarrow \mathrm{Sh}_{m,K^p}$ is étale.

Proposition 5.6 *Let h be an integer with $1 \leq h \leq 2$.*

i) *We have the following distinguished triangle:*

$$\mathcal{F}_m^{[h-1]} \rightarrow \mathcal{F}_m^{[h]} \rightarrow \mathcal{F}_m^{(h)} \rightarrow \mathcal{F}_m^{[h-1]}[1].$$

ii) *We have $\mathcal{F}_m^{(h)} = \bigoplus_{I \in \mathcal{S}_{m,h}} \mathcal{F}_{m,I}$.*

Proof. By the definition, i) is clear. ii) is also clear from Lemma 5.3 ii). ■

Proposition 5.7 *For $I \in \mathcal{S}_{m,h} \setminus \mathcal{S}_{m,h}^{\mathrm{coi}}$, we have $\mathcal{F}_{m,I}^\circ = \mathcal{F}_{m,I} = 0$.*

Proof. We will prove that $Rj_{m,I}^! R\psi \mathbb{Z}_\ell = 0$. Since the dual of $Rj_{m,I}^! R\psi \mathbb{Z}_\ell$ is isomorphic to $j_{m,I}^* R\psi \mathbb{Z}_\ell(3)[6]$, it suffices to show that, for every $x \in \overline{\mathrm{Sh}}_{m,K^p,(I)}(\overline{\mathbb{F}}_p)$, no point on the generic fiber of Sh_{m,K^p} specializes to x . In other words, for every complete discrete valuation ring R with residue field $\overline{\mathbb{F}}_p$ which is a flat \mathbb{Z}_p^∞ -algebra, and every \mathbb{Z}_p^∞ -morphism $\tilde{x}: \mathrm{Spec} R \rightarrow \mathrm{Sh}_{m,K^p}$, the image of the closed point of $\mathrm{Spec} R$ under \tilde{x} does not lie in $\overline{\mathrm{Sh}}_{m,K^p,(I)}$. This is a consequence of the following lemma. ■

Lemma 5.8 *Let R be a complete discrete valuation ring with perfect residue field k and with mixed characteristic $(0, p)$, and (X, λ) a polarized p -divisible group over R . We denote the generic (resp. special) fiber of X by X_η (resp. X_s). Then, for every $m \geq 1$, the kernel of the specialization map $X_\eta[p^m] \rightarrow X_s[p^m]$ is a coisotropic direct summand of $X_\eta[p^m]$.*

Proof. We shall prove that the kernel of the specialization map $T_p X_\eta \rightarrow T_p X_s$ is a coisotropic direct summand of $T_p X_\eta$. Consider the exact sequence $0 \rightarrow X_{s,0} \rightarrow X_s \rightarrow X_{s,\mathrm{ét}} \rightarrow 0$ over k . It is canonically lifted to the exact sequence $0 \rightarrow X_0 \rightarrow X \rightarrow X_{\mathrm{ét}} \rightarrow 0$ over R , where $X_{\mathrm{ét}}$ is an étale p -divisible group (cf. [Mes72, p. 76]). Thus we have the following commutative diagram, whose rows are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_p X_{0,\eta} & \longrightarrow & T_p X_\eta & \longrightarrow & T_p X_{\mathrm{ét},\eta} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & 0 & \longrightarrow & T_p X_s & \longrightarrow & T_p X_{s,\mathrm{ét}} \longrightarrow 0. \end{array}$$

Hence the kernel of $T_p X_\eta \rightarrow T_p X_s$ coincides with $T_p X_{0,\eta}$. Therefore it suffices to show that the composite $(T_p X_{0,\eta})^\perp \rightarrow T_p X_\eta \rightarrow T_p X_{\mathrm{ét},\eta}$ is 0.

On the other hand, by the polarization $T_p X_\eta \xrightarrow{\cong} (T_p X_\eta)^\vee(1)$, $(T_p X_{0,\eta})^\perp$ corresponds to $(T_p X_{\mathrm{ét},\eta})^\vee(1) \cong T_p X_{\mathrm{ét},\eta}^\vee$. Thus it suffices to prove that every Galois-equivariant homomorphism $T_p X_{\mathrm{ét},\eta}^\vee \rightarrow T_p X_{\mathrm{ét},\eta}$ is 0. For this, we may replace the

Tate modules $T_p X_{\text{ét},\eta}^\vee$ and $T_p X_{\text{ét},\eta}$ by the rational Tate modules $V_p X_{\text{ét},\eta}^\vee$ and $V_p X_{\text{ét},\eta}$. These are crystalline representations and the corresponding filtered φ -modules are the rational Dieudonné modules $D(X_{s,\text{ét}}^\vee)_{\mathbb{Q}}$ and $D(X_{s,\text{ét}})_{\mathbb{Q}}$, respectively. Since the slope of the former is 1 and that of the latter is 0, there is no φ -homomorphism other than 0 from $D(X_{s,\text{ét}}^\vee)_{\mathbb{Q}}$ to $D(X_{s,\text{ét}})_{\mathbb{Q}}$. This completes the proof. \blacksquare

The following corollary is immediate from Proposition 5.6 ii) and Proposition 5.7.

Corollary 5.9 *For h with $1 \leq h \leq 2$, we have $\mathcal{F}_m^{(h)} = \bigoplus_{I \in \mathcal{S}_{m,h}^{\text{coi}}} \mathcal{F}_{m,I}$.*

Let us consider the action of K_0 . Since K_0/K_m naturally acts on Sh_{m,K^p} and the action of $g \in K_0/K_m$ maps $\overline{\mathrm{Sh}}_{m,K^p,[I]}$ onto $\overline{\mathrm{Sh}}_{m,K^p,[g^{-1}I]}$, the subschemes $\overline{\mathrm{Sh}}_{m,K^p}^{[h]}$ and $\overline{\mathrm{Sh}}_{m,K^p}^{(h)}$ are preserved by the action of K_0/K_m . Therefore $\mathcal{F}_m^{[h]}$, $\mathcal{F}_m^{(h)}$, $\mathcal{F}_m^{\circ[h]}$ and $\mathcal{F}_m^{\circ(h)}$ have natural K_0/K_m -equivariant structures. Moreover, in the same way as in [Mie10a, Proposition 2.5], we can observe that $\mathcal{F}^{[h]} = \{\mathcal{F}_m^{[h]}\}_{m \geq 1}$ and $\mathcal{F}^{(h)} = \{\mathcal{F}_m^{(h)}\}_{m \geq 1}$ (resp. $\mathcal{F}^{\circ[h]} = \{\mathcal{F}_m^{\circ[h]}\}_{m \geq 1}$ and $\mathcal{F}^{\circ(h)} = \{\mathcal{F}_m^{\circ(h)}\}_{m \geq 1}$) form systems of coefficients (resp. integral coefficients) over $\{\mathcal{M}_m\}_{m \geq 1}$.

Thanks to [Mie10b], we can define J -equivariant structures on the systems of coefficients introduced above.

Proposition 5.10 *The complexes $\mathcal{F}_{m,I}^\circ$, $\mathcal{F}_m^{\circ[h]}$, $\mathcal{F}_m^{\circ(h)}$, $\mathcal{F}_{m,I}$, $\mathcal{F}_m^{[h]}$ and $\mathcal{F}_m^{(h)}$ have natural J -equivariant structures. These structures are compatible with the distinguished triangles and the direct sum decompositions in Proposition 5.6.*

Proof. We will prove the proposition for $\mathcal{F}_m^{(h)}$; other cases are similar. Put

$$\begin{aligned} \overline{\mathrm{Sh}}_{m,K^p}^{[h]\wedge} &= (\mathrm{Sh}_{m,K^p})_{/Y_{m,K^p}}^\wedge \times_{\mathrm{Sh}_{m,K^p}} \overline{\mathrm{Sh}}_{m,K^p}^{[h]}, & \overline{\mathrm{Sh}}_{m,K^p}^{(h)\wedge} &= (\overline{\mathrm{Sh}}_{m,K^p}^{[h]\wedge}, \overline{\mathrm{Sh}}_{m,K^p}^{[h-1]\wedge}), \\ \check{\mathcal{M}}_m^{[h]} &= \check{\mathcal{M}}_m \times_{(\mathrm{Sh}_{m,K^p})_{/Y_{m,K^p}}^\wedge} \overline{\mathrm{Sh}}_{m,K^p}^{[h]\wedge}, & \check{\mathcal{M}}_m^{(h)} &= (\check{\mathcal{M}}_m^{[h]}, \check{\mathcal{M}}_m^{[h-1]}). \end{aligned}$$

Then, by [Mie10b, Proposition 3.11], we have the canonical isomorphism

$$(Rj_{m*}^{(h)} Rj_m^{(h)!} R\psi \mathbb{Q}_\ell)|_{Y_{m,K^p}} \cong R\Psi_{(\mathrm{Sh}_{m,K^p})_{/Y_{m,K^p}}^\wedge, \overline{\mathrm{Sh}}_{m,K^p}^{(h)\wedge}} \mathbb{Q}_\ell.$$

Moreover, since θ_{m,K^p} is étale (cf. Corollary 4.4), by [Mie10b, Proposition 3.14], we have the canonical isomorphism

$$\mathcal{F}_m^{(h)} \cong R\Psi_{\check{\mathcal{M}}_m, \check{\mathcal{M}}_m^{(h)}} \mathbb{Q}_\ell.$$

Since the action of J on $\check{\mathcal{M}}_m$ preserves the closed formal subscheme $\check{\mathcal{M}}_{m,[I]}$ for every $I \in \mathcal{S}_m$, it also preserves the closed formal subscheme $\check{\mathcal{M}}_m^{[h]}$ for every h . Thus, by the functoriality [Mie10b, Proposition 3.7], $R\Psi_{\check{\mathcal{M}}_m, \check{\mathcal{M}}_m^{(h)}} \mathbb{Q}_\ell$ has a natural J -equivariant structure. We may import the structure into $\mathcal{F}_m^{(h)}$ by the isomorphism above.

The compatibilities with the exact sequence and the direct sum decomposition are clear from the construction (cf. [Mie10b, Remark 3.8]). \blacksquare

It is easy to see that the actions defined in the previous proposition give J -equivariant structures on the systems of (integral) coefficients $\mathcal{F}^{\circ[h]}$, $\mathcal{F}^{\circ(h)}$, $\mathcal{F}^{\circ[h]}$ and $\mathcal{F}^{\circ(h)}$. Thus we get the smooth representations $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[h]})$ and $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)})$ of $K_0 \times J$ (cf. Corollary 3.7).

Proposition 5.11 *There exists an isomorphism $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[0]}) \cong H_{\text{RZ}}^i$, which is compatible with the action of $K_0 \times J$.*

Proof. Let $m \geq 1$ be an integer and $U \in \mathcal{Q}_m$. Then, by [Mie10b, Corollary 4.40] and Proposition 4.3, we have the J -equivariant isomorphism

$$H_c^i(U, \mathcal{F}_m^{[0]}|_U) \cong H_c^i((\check{\mathcal{M}}_m)_{/U}^{\text{rig}} \otimes_{\mathbb{Q}_{p^\infty}} \bar{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell).$$

Since this isomorphism is functorial, we have $K_0 \times J$ -equivariant isomorphisms

$$\begin{aligned} H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_m^{[0]}) &\cong \varinjlim_{U \in \mathcal{Q}_m} H_c^i((\check{\mathcal{M}}_m)_{/U}^{\text{rig}} \otimes_{\mathbb{Q}_{p^\infty}} \bar{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell) \stackrel{(*)}{\cong} H_c^i(\check{\mathcal{M}}_m^{\text{rig}} \otimes_{\mathbb{Q}_{p^\infty}} \bar{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell), \\ H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[0]}) &\cong \varinjlim_m H_c^i(\check{\mathcal{M}}_m^{\text{rig}} \otimes_{\mathbb{Q}_{p^\infty}} \bar{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell) = H_{\text{RZ}}^i. \end{aligned}$$

For the isomorphy of $(*)$, we need [Hub98, Proposition 2.1 (iv)] and [Mie10b, Lemma 4.14]. ■

Remark 5.12 We can deduce from Proposition 5.11 and Corollary 3.7 that the action of $K_0 \times J$ on H_{RZ}^i is smooth.

5.2 G -action on $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[h]})$, $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)})$

In this subsection, we define actions of G on $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[h]})$ and $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)})$ by using the method in [Man05, §6]. Put $G^+ = \{g \in G \mid g^{-1}L \subset L\}$, which is a submonoid of G . For $g \in G^+$, let $e(g)$ be the minimal non-negative integer such that $\text{Ker}(g^{-1}: V/L \rightarrow V/L)$ is contained in $p^{-e(g)}L/L$. Since $\text{Ker } g^{-1} = (gL+L)/L$, we have $gL \subset p^{-e(g)}L$.

In the sequel, we fix a compact open subgroup K^p of $\text{GSp}(V'_{\mathbb{A}^\infty, p})$ and denote Sh_{m, K^p} , $\bar{\text{Sh}}_{m, K^p}$, $\bar{\text{Sh}}_{m, K^p, [I]}$, \dots by Sh_m , $\bar{\text{Sh}}_m$, $\bar{\text{Sh}}_{m, [I]}$, \dots , respectively. Moreover, we fix $g \in G^+$ and denote $e(g)$ by e for simplicity.

Assume that $m \geq e$. Let us consider the \mathbb{Z}_{p^∞} -scheme $\text{Sh}_{m, g}$ such that for a \mathbb{Z}_{p^∞} -scheme S , the set $\text{Sh}_{m, g}(S)$ consists of isomorphism classes of quintuples $(A, \lambda, \eta^p, \eta_p, \mathcal{E})$ satisfying the following.

- The quadruple $(A, \lambda, \eta^p, \eta_p)$ gives an element of $\text{Sh}_m(S)$.
- $\mathcal{E} \subset X[p^e]$ is a finite flat subgroup scheme of order $p^{vp(\det g^{-1})}$, where we put $X = A[p^\infty]$. It is self-dual with respect to λ , and satisfies $\eta'_p(\text{Ker } g^{-1}) \subset \mathcal{E}(S)$, where η'_p denotes the composite $p^{-m}L/L \xrightarrow{\times p^m} L/p^mL \xrightarrow{\eta_p} X[p^m]$.

– For \mathcal{E} as above, we have the following commutative diagram:

$$\begin{array}{ccccc}
 p^{-m}L/L & \xrightarrow{\eta'_p} & X[p^m] & \hookrightarrow & X \\
 g^{-1} \downarrow & & \downarrow & & \downarrow \\
 p^{-m}g^{-1}L/L & \longrightarrow & X[p^m]/\mathcal{E} & \hookrightarrow & X/\mathcal{E} \\
 \uparrow & & \uparrow & & \uparrow \\
 L/p^{m-e}L & \xrightarrow{\cong} & p^{-m+e}L/L & \longrightarrow & (X/\mathcal{E})[p^{m-e}].
 \end{array}$$

We denote the composite of the lowest row by $\eta_p \circ g$ and assume that it gives a Drinfeld $(m-e)$ -level structure.

We have the two natural morphisms

$$\begin{aligned}
 \mathrm{pr}: \mathrm{Sh}_{m,g} &\longrightarrow \mathrm{Sh}_m; (A, \lambda, \eta^p, \eta_p, \mathcal{E}) \longmapsto (A, \lambda, \eta^p, \eta_p), \\
 [g]: \mathrm{Sh}_{m,g} &\longrightarrow \mathrm{Sh}_{m-e}; (A, \lambda, \eta^p, \eta_p, \mathcal{E}) \longmapsto (A/\mathcal{E}, \lambda, \eta^p, \eta_p \circ g).
 \end{aligned}$$

It is known that these are proper morphisms, pr induces an isomorphism on the generic fibers, and $[g]$ induces the action of g on the generic fibers [Man05, Proposition 16, Proposition 17].

We can easily see that $\{\mathrm{Sh}_{m,g}\}_{m \geq e}$ form a projective system whose transition maps are finite. Obviously, pr and $[g]$ are compatible with change of m .

Similarly we can define the formal scheme $\check{\mathcal{M}}_{m,g}$ and the morphisms $\mathrm{pr}: \check{\mathcal{M}}_{m,g} \longrightarrow \check{\mathcal{M}}_m$ and $[g]: \check{\mathcal{M}}_{m,g} \longrightarrow \check{\mathcal{M}}_{m-e}$. The former morphism induces an isomorphism on the Raynaud generic fibers and the composite $[g]^{\mathrm{rig}} \circ (\mathrm{pr}^{\mathrm{rig}})^{-1}$ coincides with the action of g . The group J naturally acts on $\check{\mathcal{M}}_{m,g}$ and two morphisms pr and $[g]$ are compatible with the action of J . Moreover, if we denote by $Y_{m,g}$ the inverse image of $Y_m \subset \mathrm{Sh}_m$ under $\mathrm{pr}: \mathrm{Sh}_{m,g} \longrightarrow \mathrm{Sh}_m$, then we can construct a morphism $\theta_{m,g}: \check{\mathcal{M}}_{m,g} \longrightarrow (\mathrm{Sh}_{m,g})_{Y_{m,g}}^\wedge$ which makes the following diagrams cartesian:

$$\begin{array}{ccc}
 \check{\mathcal{M}}_{m,g} & \xrightarrow{\theta_{m,g}} & (\mathrm{Sh}_{m,g})_{Y_{m,g}}^\wedge \\
 \downarrow \mathrm{pr} & & \downarrow \mathrm{pr} \\
 \check{\mathcal{M}}_m & \xrightarrow{\theta_m} & (\mathrm{Sh}_m)_{Y_m}^\wedge
 \end{array}
 \quad
 \begin{array}{ccc}
 \check{\mathcal{M}}_{m,g} & \xrightarrow{\theta_{m,g}} & (\mathrm{Sh}_{m,g})_{Y_{m,g}}^\wedge \\
 \downarrow [g] & & \downarrow [g] \\
 \check{\mathcal{M}}_{m-e} & \xrightarrow{\theta_{m-e}} & (\mathrm{Sh}_{m-e})_{Y_{m-e}}^\wedge.
 \end{array}$$

Now let h be an integer with $1 \leq h \leq 2$ and $I \in \mathcal{S}_{m,h}$. Then we can define the subschemes $\overline{\mathrm{Sh}}_{m,g,[I]}$, $\overline{\mathrm{Sh}}_{m,g,(I)}$, $\overline{\mathrm{Sh}}_{m,g}^{[h]}$ and $\overline{\mathrm{Sh}}_{m,g}^{(h)}$ of $\mathrm{Sh}_{m,g}$ in the same way as $\overline{\mathrm{Sh}}_{m,[I]}$, $\overline{\mathrm{Sh}}_{m,(I)}$, $\overline{\mathrm{Sh}}_m^{[h]}$ and $\overline{\mathrm{Sh}}_m^{(h)}$. The following proposition is obvious:

Proposition 5.13 *We have the commutative diagrams below:*

$$\begin{array}{ccccc}
 \overline{\mathrm{Sh}}_{m,g,(I)} & \longrightarrow & \overline{\mathrm{Sh}}_{m,g,[I]} & \longrightarrow & \overline{\mathrm{Sh}}_{m,g} \\
 \downarrow & & \downarrow & & \downarrow \mathrm{pr} \\
 \overline{\mathrm{Sh}}_{m,(I)} & \longrightarrow & \overline{\mathrm{Sh}}_{m,[I]} & \longrightarrow & \overline{\mathrm{Sh}}_m
 \end{array}
 \quad
 \begin{array}{ccccc}
 \overline{\mathrm{Sh}}_{m,g}^{(h)} & \longrightarrow & \overline{\mathrm{Sh}}_{m,g}^{[h]} & \longrightarrow & \overline{\mathrm{Sh}}_{m,g} \\
 \downarrow & & \downarrow & & \downarrow \mathrm{pr} \\
 \overline{\mathrm{Sh}}_m^{(h)} & \longrightarrow & \overline{\mathrm{Sh}}_m^{[h]} & \longrightarrow & \overline{\mathrm{Sh}}_m.
 \end{array}$$

The rectangles in the left diagram is cartesian. The rectangles in the right diagram is cartesian up to nilpotent elements (namely, $\overline{\text{Sh}}_{m,g}^{[h]} \longrightarrow \overline{\text{Sh}}_m^{[h]} \times_{\overline{\text{Sh}}_m} \overline{\text{Sh}}_{m,g}$ induces a homeomorphism on the underlying topological spaces, and so on).

Let us consider how $\overline{\text{Sh}}_{m,g,[I]}$ are mapped by $[g]: \text{Sh}_{m,g} \longrightarrow \text{Sh}_{m-e}$. For this purpose, let us introduce some notation.

Definition 5.14 We denote by $\mathcal{S}_{\infty,h}$ the set of direct summands of L of rank $4-h$ and by $\mathcal{S}_{\infty,h}^{\text{coi}}$ the subset of $\mathcal{S}_{\infty,h}$ consisting of coisotropic direct summands. We can identify $\mathcal{S}_{\infty,h}$ with the set of direct summands of V of rank $4-h$; thus G naturally acts on $\mathcal{S}_{\infty,h}$ and $\mathcal{S}_{\infty,h}^{\text{coi}}$. Let $g^{-1}: \mathcal{S}_{m,h} \longrightarrow \mathcal{S}_{m-e,h}$ be the unique map which makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{S}_{\infty,h} & \longrightarrow & \mathcal{S}_{m,h} \\ \downarrow g^{-1} & & \downarrow g^{-1} \\ \mathcal{S}_{\infty,h} & \longrightarrow & \mathcal{S}_{m-e,h}. \end{array}$$

The existence of such g^{-1} follows from $p^m L \subset p^e L \subset g^{-1} L \subset L$. Indeed, for direct summands I, I' of V , we have

$$\begin{aligned} I \cap L + p^m L = I' \cap L + p^m L &\implies g^{-1} I \cap g^{-1} L + p^m L = g^{-1} I' \cap g^{-1} L + p^m L \\ &\implies g^{-1} I \cap g^{-1} L \cap p^e L + p^m L = g^{-1} I' \cap g^{-1} L \cap p^e L + p^m L \\ &\iff g^{-1} I \cap p^e L + p^m L = g^{-1} I' \cap p^e L + p^m L \\ &\iff g^{-1} I \cap L + p^{m-e} L = g^{-1} I' \cap L + p^{m-e} L. \end{aligned}$$

Obviously $g^{-1}: \mathcal{S}_{m,h} \longrightarrow \mathcal{S}_{m-e,h}$ induces a map from $\mathcal{S}_{m,h}^{\text{coi}}$ to $\mathcal{S}_{m-e,h}^{\text{coi}}$.

Proposition 5.15 i) For $h \in \{1, 2\}$ and $I \in \mathcal{S}_{m,h}$, $[g]$ induces morphisms

$$\begin{aligned} \text{Sh}_{m,g,[I]} &\longrightarrow \text{Sh}_{m-e,[g^{-1}I]}, & \text{Sh}_{m,g,(I)} &\longrightarrow \text{Sh}_{m-e,(g^{-1}I)}, \\ \text{Sh}_{m,g}^{[h]} &\longrightarrow \text{Sh}_{m-e}^{[h]}, & \text{Sh}_{m,g}^{(h)} &\longrightarrow \text{Sh}_{m-e}^{(h)}. \end{aligned}$$

ii) The rectangles of the following commutative diagram is cartesian up to nilpotent elements:

$$\begin{array}{ccccc} \overline{\text{Sh}}_{m,g}^{(h)} & \longrightarrow & \overline{\text{Sh}}_{m,g}^{[h]} & \longrightarrow & \overline{\text{Sh}}_{m,g} \\ \downarrow & & \downarrow & & \downarrow [g] \\ \overline{\text{Sh}}_{m-e}^{(h)} & \longrightarrow & \overline{\text{Sh}}_{m-e}^{[h]} & \longrightarrow & \overline{\text{Sh}}_{m-e}. \end{array}$$

Proof. By the definition of $[g]$, it is clear that $[g]$ induces a morphism $\text{Sh}_{m,g,[I]} \longrightarrow \text{Sh}_{m-e,[g^{-1}I]}$ for $I \in \mathcal{S}_{m,h}$, and thus induces a morphism $\text{Sh}_{m,g}^{[h]} \longrightarrow \text{Sh}_{m-e}^{[h]}$. On the other hand, note that, for every $(A, \lambda, \eta^p, \eta_p, \mathcal{E}) \in \text{Sh}_{m,g}(\overline{\mathbb{F}}_p)$, the p -divisible groups $A[p^\infty]$ and $A[p^\infty]/\mathcal{E}$ are isogenous, and thus have the same étale heights.

Therefore, by Lemma 5.3 i), the inverse image of $\overline{\mathrm{Sh}}_{m-e}^{[h]}$ (resp. $\overline{\mathrm{Sh}}_{m-e}^{(h)}$) under $[g]$ coincides with $\overline{\mathrm{Sh}}_{m,g}^{[h]}$ (resp. $\overline{\mathrm{Sh}}_{m,g}^{(h)}$) as sets. Therefore a morphism $\mathrm{Sh}_{m,g}^{(h)} \rightarrow \mathrm{Sh}_{m-e}^{(h)}$ is naturally induced and the rectangles in the diagram above are cartesian up to nilpotent elements. Finally, since $\overline{\mathrm{Sh}}_{m,g,(I)} = \overline{\mathrm{Sh}}_{m,g,[I]} \cap \overline{\mathrm{Sh}}_{m,g}^{(h)}$ and $\overline{\mathrm{Sh}}_{m-e,(g^{-1}I)} = \overline{\mathrm{Sh}}_{m-e,[g^{-1}I]} \cap \overline{\mathrm{Sh}}_{m-e}^{(h)}$, $[g]$ induces a morphism $\overline{\mathrm{Sh}}_{m,g,(I)} \rightarrow \overline{\mathrm{Sh}}_{m-e,(g^{-1}I)}$. \blacksquare

By Proposition 5.13 and Proposition 5.15, we have the natural cohomological correspondence γ_g from $\mathcal{F}_{m-e}^{[h]}$ (resp. $\mathcal{F}_{m-e}^{(h)}$) to $\mathcal{F}_m^{[h]}$ (resp. $\mathcal{F}_m^{(h)}$); see §6. This cohomological correspondence induces a homomorphism γ_g from $H_c^i(\bar{\mathcal{M}}_{m-e}, \mathcal{F}_{m-e}^{[h]})$ (resp. $H_c^i(\bar{\mathcal{M}}_{m-e}, \mathcal{F}_{m-e}^{(h)})$) to $H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_m^{[h]})$ (resp. $H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_m^{(h)})$). Indeed, for $U \in \mathcal{Q}_{m-e}$, we can take $U' \in \mathcal{Q}_m$ which contains $\mathrm{pr}([g]^{-1}(U))$. Then γ_g induces $H_c^i(U, \mathcal{F}_{m-e}^{[h]}|_U) \rightarrow H_c^i(U', \mathcal{F}_m^{[h]}|_{U'})$, and therefore induces $H_c^i(\bar{\mathcal{M}}_{m-e}, \mathcal{F}_{m-e}^{[h]}) \rightarrow H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_m^{[h]})$. It is easy to see that this homomorphism is compatible with change of m ; hence we get the endomorphism γ_g on $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[h]})$ and $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)})$.

Lemma 5.16 *The endomorphism γ_g commutes with the action of J on $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[h]})$ and $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)})$.*

Proof. We will only consider γ_g on $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[h]})$, since the other case is similar. Let $U \in \mathcal{Q}_{m-e}$ and $U' \in \mathcal{Q}_m$ be as above and put $W = [g]^{-1}(U)$, $W' = \mathrm{pr}^{-1}(U')$. It suffices to show the commutativity of the following diagram for $j \in J$:

$$\begin{array}{ccccccc} H_c^i(jU, \mathcal{F}_{m-e}^{[h]}|_{jU}) & \xrightarrow{[g]^*} & H_c^i(jW, \mathcal{F}_{m-e}^{[h]}|_{jW}) & \rightarrow & H_c^i(jW', \mathcal{F}_{m-e}^{[h]}|_{jW'}) & \xrightarrow{\mathrm{pr}_*} & H_c^i(jU', \mathcal{F}_{m-e}^{[h]}|_{jU'}) \\ \downarrow j & & \downarrow j & & \downarrow j & & \downarrow j \\ H_c^i(U, \mathcal{F}_{m-e}^{[h]}|_U) & \xrightarrow{[g]^*} & H_c^i(W, \mathcal{F}_{m-e}^{[h]}|_W) & \longrightarrow & H_c^i(W', \mathcal{F}_{m-e}^{[h]}|_{W'}) & \xrightarrow{\mathrm{pr}_*} & H_c^i(U', \mathcal{F}_{m-e}^{[h]}|_{U'}) \end{array}$$

By the construction of the J -actions, the left and the middle rectangles are commutative. On the other hand, since pr is proper and induces an isomorphism on the generic fiber, pr_* is an isomorphism and its inverse is pr^* . As pr^* commutes with the J -action, the right rectangle above is also commutative. This concludes the proof. \blacksquare

Lemma 5.17 i) For $g, g' \in G^+$, $\gamma_{gg'} = \gamma_g \circ \gamma_{g'}$.

ii) For $g \in K_0$, γ_g coincides with the action of K_0 on $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[h]})$ or $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)})$, which we already introduced.

iii) The endomorphism $\gamma_{p^{-1} \cdot \mathrm{id}}$ is an isomorphism (in fact, it coincides with the action of $p^{-1} \cdot \mathrm{id} \in J$).

Proof. i) follows from Corollary 6.3. ii) and iii) are consequences of [Man05, Proposition 16, Proposition 17] and the analogous properties for the Rapoport-Zink spaces (cf. [Man04, Proposition 7.4 (4), (5)]). \blacksquare

Note that G is generated by G^+ and $p \cdot \text{id}$ as a monoid. Therefore, by the lemma above, we can extend the actions of K_0 on $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[h]})$ and $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)})$ to whole G . Together with Lemma 5.16, we have a smooth $G \times J$ -module structures on $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[h]})$ and $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)})$. We can observe without difficulty that the isomorphism in Proposition 5.11 is in fact compatible with the action of G :

Proposition 5.18 *The isomorphism $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[0]}) \cong H_{\text{RZ}}^i$ in Proposition 5.11 is an isomorphism of $G \times J$ -modules.*

Next we investigate the G -module structure of $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)})$ for $h \in \{1, 2\}$. Let us fix an element $\tilde{I}(h)$ of $\mathcal{S}_{\infty, h}^{\text{coi}}$ and denote its image under the natural map $\mathcal{S}_{\infty, h}^{\text{coi}} \rightarrow \mathcal{S}_{m, h}^{\text{coi}}$ by $\tilde{I}(h)_m$. Put $P_h = \text{Stab}_G(\tilde{I}(h))$, which is a maximal parabolic subgroup of G . Then we can identify $\mathcal{S}_{\infty, h}$ with $G/P_h = K_0/(P_h \cap K_0)$ and $\mathcal{S}_{m, h}$ with $K_m \backslash G/P_h = K_m \backslash K_0/(P_h \cap K_0)$. For $g \in G^+$ and an integer m with $m \geq e := e(g)$, $g^{-1}: \mathcal{S}_{m, h} \rightarrow \mathcal{S}_{m-e, h}$ is identified with the map $K_m \backslash G/P_h \rightarrow K_{m-e} \backslash G/P_h$; $K_m x P_h \mapsto K_{m-e} g^{-1} x P_h$.

Definition 5.19 We put $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}_{\tilde{I}(h)}) = \varinjlim_m H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_{m, \tilde{I}(h)_m})$. Here the transition maps are given as follows: for integers $1 \leq m \leq m'$,

$$\begin{aligned} H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_{m, \tilde{I}(h)_m}) &\longrightarrow H_c^i(\bar{\mathcal{M}}_{m'}, p_{mm'}^* \mathcal{F}_{m, \tilde{I}(h)_m}) \longrightarrow \bigoplus_{\substack{I' \in \mathcal{S}_{m', h}^{\text{coi}} \\ I'/p^m I' = \tilde{I}(h)_m}} H_c^i(\bar{\mathcal{M}}_{m'}, \mathcal{F}_{m', I'}) \\ &\longrightarrow H_c^i(\bar{\mathcal{M}}_{m'}, \mathcal{F}_{m', \tilde{I}(h)_{m'}}). \end{aligned}$$

It is easy to see that $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}_{\tilde{I}(h)})$ has a structure of a smooth $P_h \times J$ -module (use Theorem 3.4 and Proposition 5.15 i)). For each $m \geq 1$ we have the homomorphism

$$H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_m^{(h)}) = \bigoplus_{I \in \mathcal{S}_{m, h}^{\text{coi}}} H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_{m, I}) \longrightarrow H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_{m, \tilde{I}(h)_m}),$$

which induces the homomorphism $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)}) \rightarrow H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}_{\tilde{I}(h)})$. By Proposition 5.15 i), we can prove that this is a homomorphism of $P_h \times J$ -modules.

Proposition 5.20 *We have an isomorphism $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)}) \cong \text{Ind}_{P_h}^G H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}_{\tilde{I}(h)})$ of $G \times J$ -modules.*

Proof. By the Frobenius reciprocity, we have a G -homomorphism $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)}) \rightarrow \text{Ind}_{P_h}^G H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}_{\tilde{I}(h)})$. We shall observe that this is bijective. For an integer $m \geq 1$, we have

$$\begin{aligned} H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_m^{(h)}) &= \bigoplus_{I \in \mathcal{S}_{m, h}^{\text{coi}}} H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_{m, I}) = \bigoplus_{g \in K_m \backslash K_0 / (P_h \cap K_0)} H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_{m, g^{-1} \tilde{I}(h)_m}) \\ &\cong \text{Ind}_{(P_h \cap K_0) / (P_h \cap K_m)}^{K_0 / K_m} H_c^i(\bar{\mathcal{M}}_m, \mathcal{F}_{m, \tilde{I}(h)_m}), \end{aligned}$$

where the last isomorphism, due to [Boy99, Lemme 13.2], is an isomorphism as K_0 -modules. By taking the inductive limit, we have isomorphisms

$$H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)}) \xrightarrow{\cong} \mathrm{Ind}_{P_h \cap K_0}^{K_0} H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}_{\tilde{I}(h)_m}) \xleftarrow{\cong} \mathrm{Ind}_{P_h}^G H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}_{\tilde{I}(h)_m})$$

(the second isomorphism follows from the Iwasawa decomposition $G = P_h K_0$). By the proof of [Boy99, Lemme 13.2], it is easy to see that the first isomorphism above is nothing but the K_0 -homomorphism obtained by the Frobenius reciprocity for $P_h \cap K_0 \subset K_0$. Therefore the composite of the two isomorphisms above coincides with the G -homomorphism introduced at the beginning of this proof. Thus we conclude the proof. \blacksquare

5.3 Proof of the main theorem

We begin with the following result on non-cuspidality:

Theorem 5.21 *For every $i \in \mathbb{Z}$ and $h \in \{1, 2\}$, the G -module $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)})_{\overline{\mathbb{Q}}_\ell}$ has no quasi-cuspidal subquotient.*

By Proposition 5.20 and [Bern84, 2.4], it suffices to show the following proposition:

Proposition 5.22 *Let $h \in \{1, 2\}$. The unipotent radical U_h of P_h acts trivially on $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}_{\tilde{I}(h)})_{\overline{\mathbb{Q}}_\ell}$.*

To prove Proposition 5.22, we need some preparations. In the sequel, let \mathbf{G} and \mathbf{H} be connected reductive groups over \mathbb{Q}_p , \mathbf{P} a parabolic subgroup of \mathbf{G} and \mathbf{U} the unipotent radical of \mathbf{P} . We put $P = \mathbf{P}(\mathbb{Q}_p)$, $H = \mathbf{H}(\mathbb{Q}_p)$ and $U = \mathbf{U}(\mathbb{Q}_p)$.

Lemma 5.23 *Let A be a noetherian \mathbb{Q} -algebra and V an A -module with a smooth P -action. Assume that V is A -admissible in the sense of [Bern84, 1.16]. Then U acts on V trivially.*

Proof. First assume that A is Artinian. Then we can prove the lemma in the same way as [Boy99, Lemme 13.2.3] (we use length in place of dimension).

For the general case, we use noetherian induction. Assume that the lemma holds for every proper quotient of A . Take a minimal prime ideal \mathfrak{p} of A . Then $A_{\mathfrak{p}}$ is Artinian and $V_{\mathfrak{p}}$ is an $A_{\mathfrak{p}}$ -admissible representation of P (note that $(V_{\mathfrak{p}})^K = (V^K)_{\mathfrak{p}}$ for every compact open subgroup K of P). Therefore U acts on $V_{\mathfrak{p}}$ trivially. Let V' (resp. V'') be the kernel (resp. image) of $V \rightarrow V_{\mathfrak{p}}$. Note that V' and V'' are A -admissible representations of P , for A is noetherian.

Consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' \longrightarrow 0 \\ & & \downarrow (1) & & \downarrow (2) & & \downarrow (3) \\ 0 & \longrightarrow & V'_U & \longrightarrow & V_U & \longrightarrow & (V_{\mathfrak{p}})_U. \end{array}$$

It is well-known that the functor taking U -coinvariant $V \mapsto V_U$ is an exact functor; thus the lower row in the diagram above is exact. On the other hand, the arrow labeled (3) is injective, since it is the composite of $V'' \hookrightarrow V_{\mathfrak{p}} \xrightarrow{\cong} (V_{\mathfrak{p}})_U$. Therefore, by the snake lemma, the injectivity of (2) is equivalent to that of (1). In other words, we have only to prove that the action of U on V' is trivial.

On the other hand, by the definition, V' is the union of $V_s := \{x \in V \mid sx = 0\}$ for $s \in A \setminus \mathfrak{p}$. Since V_s can be regarded as an admissible $A/(s)$ -representation, U acts on V_s trivially by the induction hypothesis. Hence U acts on V' trivially. \blacksquare

Proposition 5.24 *Let V be a smooth representation of $P \times H$ over $\overline{\mathbb{Q}}_\ell$ and assume that for every compact open subgroup K of P , V^K is a finitely generated H -module. Then U acts on V trivially.*

Proof. Since $\overline{\mathbb{Q}}_\ell$ and \mathbb{C} are isomorphic as fields, we may replace $\overline{\mathbb{Q}}_\ell$ in the statement by \mathbb{C} . Let \mathfrak{Z} be the Bernstein center of H [Bern84]. It is decomposed as $\mathfrak{Z} = \prod_{\theta \in \Theta} \mathfrak{Z}_\theta$, where Θ denotes the set of connected components of the Bernstein variety of H . For $\theta \in \Theta$, we denote the θ -part of V by V_θ . Then we have the canonical decomposition $V = \bigoplus_{\theta \in \Theta} V_\theta$, which is compatible with the action of $P \times H$. Therefore, by replacing V with V_θ , we may assume that the action of \mathfrak{Z} on V factors through \mathfrak{Z}_θ for some $\theta \in \Theta$.

By the assumption and [Bern84, Proposition 3.3], for every compact open subgroup K of P , V^K is a \mathfrak{Z}_θ -admissible H -module. Namely, for every compact open subgroup K (resp. K') of P (resp. H), $V^{K \times K'}$ is a finitely generated \mathfrak{Z}_θ -module. In other words, for every compact open subgroup K' of H , $V^{K'}$ is a \mathfrak{Z}_θ -admissible P -module. Since \mathfrak{Z}_θ is a finitely generated \mathbb{C} -algebra, U acts trivially on $V^{K'}$ by Lemma 5.23. Therefore U acts trivially on V also. \blacksquare

Proof of Proposition 5.22. By Proposition 5.24, we have only to prove that, for every $m \geq 1$, $H_c^i(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)})^{P_h \cap K_m}$ is a finitely generated J -module (recall that a finitely generated J -module is noetherian [Bern84, Remarque 3.12]). As a J -module, it is a direct summand of $(\text{Ind}_{P_h}^G H_c^i(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)}))^{K_m} \cong H_c^i(\mathcal{M}_\infty, \mathcal{F}^{(h)})^{K_m}$. On the other hand, by Corollary 3.7, $H_c^i(\mathcal{M}_\infty, \mathcal{F}^{(h)})^{K_m}$ is a finitely generated J -module. Thus $H_c^i(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)})^{P_h \cap K_m}$ is also finitely generated. \blacksquare

Proposition 5.25 *Let i be an integer. If $i \geq 5$, then $H_c^i(\mathcal{M}_\infty, \mathcal{F}^{[2]}) = 0$. On the other hand, if $i \leq 1$, then $H_c^i(\mathcal{M}_\infty, \mathcal{F}^{[0]}) = 0$.*

Proof. By the definition, it suffices to show that for every $m \geq 1$ and every $U \in \mathcal{Q}_m$ we have $H_c^i(U, \mathcal{F}_m^{[2]}|_U) = 0$ for $i \geq 5$ and $H_c^i(U, \mathcal{F}_m^{[0]}|_U) = 0$ for $i \leq 1$. Thus the claim is reduced to the following lemma. \blacksquare

Lemma 5.26 *Let S be the spectrum of a strict henselian discrete valuation ring and X a separated S -scheme of finite type. We denote its special (resp. generic)*

fiber by X_s (resp. X_η). Let Z be a closed subscheme of X_s and denote the natural closed immersion $Z \hookrightarrow X$ by i . Assume that X_η is smooth of pure dimension d and Z is purely d' -dimensional.

Then we have $H^n(Z, i^* R\psi_X \mathbb{Q}_\ell) = H_c^n(Z, i^* R\psi_X \mathbb{Q}_\ell) = 0$ for $n > d + d'$ and $H_c^n(Z, Ri^! R\psi_X \mathbb{Q}_\ell) = 0$ for $n < d - d'$.

Proof. First note that $H^n(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = H_c^n(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = 0$ if $n > 2 \dim Z$ or $n > 2 \dim(\mathrm{supp} R^k \psi_X \mathbb{Q}_\ell)$. By [BBD82, Proposition 4.4.2], for each $k \geq 0$ we have $\dim(\mathrm{supp} R^k \psi_X \mathbb{Q}_\ell) \leq d - k$; therefore if $n + k > d + d'$ then we have

$$\begin{aligned} n &> d' + (d - k) \geq \dim Z + \dim(\mathrm{supp} R^k \psi_X \mathbb{Q}_\ell) \\ &\geq 2 \min\{\dim Z, \dim(\mathrm{supp} R^k \psi_X \mathbb{Q}_\ell)\} \end{aligned}$$

and thus $H^n(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = H_c^n(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = 0$. By the spectral sequence, we have $H^n(Z, i^* R\psi_X \mathbb{Q}_\ell) = H_c^n(Z, i^* R\psi_X \mathbb{Q}_\ell) = 0$ for $n > d + d'$.

On the other hand, by the Poincaré duality, we have

$$\begin{aligned} H_c^n(Z, Ri^! R\psi_X \mathbb{Q}_\ell) &= H^{-n}(Z, D_Z(Ri^! R\psi_X \mathbb{Q}_\ell))^\vee = H^{-n}(Z, i^* R\psi_X D_{X_\eta} \mathbb{Q}_\ell)^\vee \\ &= H^{-n}(Z, i^* R\psi_X \mathbb{Q}_\ell(d)[2d])^\vee = H^{2d-n}(Z, i^* R\psi_X \mathbb{Q}_\ell)^\vee(-d), \end{aligned}$$

where D_Z (resp. D_{X_η}) denotes the dualizing functor with respect to Z (resp. X_η). Therefore it vanishes if $2d - n > d + d'$, namely, $n < d - d'$. \blacksquare

Now we can prove our main theorem.

Proof of Theorem 3.2. By Proposition 5.11 and Proposition 5.25, we have $H_{\mathrm{RZ}}^i = 0$ for $i \leq 1$. Therefore we may assume that $i \geq 5$.

By Proposition 5.6 i), we have the exact sequence of smooth G -modules

$$H_c^{i-1}(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)})_{\overline{\mathbb{Q}}_\ell} \longrightarrow H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[h-1]})_{\overline{\mathbb{Q}}_\ell} \longrightarrow H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[h]})_{\overline{\mathbb{Q}}_\ell}$$

for every h with $1 \leq h \leq 2$. Moreover, $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)})_{\overline{\mathbb{Q}}_\ell}$ has no quasi-cuspidal subquotient by Theorem 5.21. Thus, starting from $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[2]})_{\overline{\mathbb{Q}}_\ell} = 0$ (Proposition 5.25), we can inductively prove that $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[h]})_{\overline{\mathbb{Q}}_\ell}$ has no quasi-cuspidal subquotient; indeed, the property that a representation has no quasi-cuspidal subquotient is stable under sub, quotient and extension (use the canonical decomposition in [Bern84, 2.3.1]). In particular, $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[0]})_{\overline{\mathbb{Q}}_\ell} \cong H_{\mathrm{RZ}, \overline{\mathbb{Q}}_\ell}^i$ (cf. Proposition 5.18) has no quasi-cuspidal subquotient. This completes the proof. \blacksquare

6 Appendix: Complements on cohomological correspondences

In this section, we recall the notion of cohomological correspondences (cf. [SGA5, Exposé III], [Fuj97]) and give some results on them. These are used to define the action of G on $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{[h]})$ and $H_c^i(\bar{\mathcal{M}}_\infty, \mathcal{F}^{(h)})$.

In this section, we change our notation. Let k be a field and ℓ a prime number which is invertible in k . We denote one of $\mathbb{Z}/\ell^n\mathbb{Z}$ or \mathbb{Q}_ℓ by Λ . Let X_1 and X_2 be schemes which are separated of finite type over k , and L_i an object of $D_c^b(X_i, \Lambda)$ for $i = 1, 2$ respectively. A *cohomological correspondence* from L_1 to L_2 is a pair (γ, c) consisting of a separated k -morphism of finite type $\gamma: \Gamma \rightarrow X_1 \times X_2$ and a morphism $c: \gamma_1^* L_1 \rightarrow R\gamma_2^! L_2$ in the category $D_c^b(\Gamma, \Lambda)$, where we denote $\text{pr}_i \circ \gamma$ by γ_i . For simplicity, we also write c for (γ, c) , if there is no risk of confusion. If we are given a cohomological correspondence (γ, c) where γ_1 is proper, then we have the associated morphism $R\Gamma_c(c): R\Gamma_c(X_1, L_1) \rightarrow R\Gamma_c(X_2, L_2)$ by composing

$$\begin{aligned} R\Gamma_c(X_1, L_1) &\xrightarrow{\gamma_1^*} R\Gamma_c(\Gamma, \gamma_1^* L_1) \xrightarrow{R\Gamma_c(c)} R\Gamma_c(\Gamma, R\gamma_2^! L_2) = R\Gamma_c(X_2, R\gamma_{2!} R\gamma_2^! L_2) \\ &\xrightarrow{\text{adj}} R\Gamma_c(X_2, L_2). \end{aligned}$$

We can compose two cohomological correspondences. Let X_3 be another scheme which is separated of finite type over k and $L_3 \in D_c^b(X_3, \Lambda)$. Let (γ', c') be a cohomological correspondence from L_2 to L_3 . Consider the following diagram

$$\begin{array}{ccccc} \Gamma \times_{X_2} \Gamma' & \xrightarrow{\text{pr}_2} & \Gamma' & \xrightarrow{\gamma'_2} & X_3 \\ \downarrow \text{pr}_1 & & \downarrow \gamma'_1 & & \\ \Gamma & \xrightarrow{\gamma_2} & X_2 & & \\ \downarrow \gamma_1 & & & & \\ X_1 & & & & \end{array}$$

Let γ'' be the natural morphism $\Gamma \times_{X_2} \Gamma' \rightarrow X_1 \times X_3$ and $c'': \gamma_1''^* L_1 \rightarrow R\gamma_2''^! L_3$ the composite of

$$\gamma_1''^* L_1 = \text{pr}_1^* \gamma_1^* L_1 \xrightarrow{\text{pr}_1^*(c)} \text{pr}_1^* R\gamma_2^! L_2 \xrightarrow{\text{b.c.}} R\text{pr}_2^! \gamma_1'^* L_2 \xrightarrow{R\text{pr}_2^!(c')} R\text{pr}_2^! R\gamma_2'^! L_3 = R\gamma_2''^! L_3,$$

where b.c. denotes the base change morphism. We call the cohomological correspondence (γ'', c'') the composite of (γ, c) and (γ', c') , and denote it by $c' \circ c$. It is not difficult to see that if γ_1 and γ'_1 are proper, then γ_1'' is also proper and $R\Gamma_c(c' \circ c) = R\Gamma_c(c') \circ R\Gamma_c(c)$.

Let us recall some operations for cohomological correspondences. Let X_1, X_2, X'_1 and X'_2 be schemes which are separated of finite type over k , and $\gamma: \Gamma \rightarrow X_1 \times X_2$ and $\gamma': \Gamma' \rightarrow X'_1 \times X'_2$ separated k -morphisms of finite type. Assume that the following commutative diagram is given:

$$\begin{array}{ccccc} X'_1 & \xleftarrow{\gamma'_1} & \Gamma' & \xrightarrow{\gamma'_2} & X'_2 \\ \downarrow a_1 & & \downarrow a & & \downarrow a_2 \\ X_1 & \xleftarrow{\gamma_1} & \Gamma & \xrightarrow{\gamma_2} & X_2. \end{array}$$

First assume that every vertical morphism is proper. Let L'_i be an object of $D_c^b(X'_i, \Lambda)$ for each $i = 1, 2$ and (γ', c') a cohomological correspondence from L'_1 to L'_2 . Then we can define the cohomological correspondence (γ, a_*c') from $Ra_{1*}L'_1$ to $Ra_{2*}L'_2$ by

$$\begin{aligned} \gamma_1^* Ra_{1*} L'_1 &\xrightarrow{\text{b.c.}} Ra_* \gamma_1'^* L'_1 \xrightarrow{Ra_*(c')} Ra_* R\gamma_2'^! L'_2 = Ra_* R\gamma_2'^! L'_2 \\ &\xrightarrow{\text{b.c.}} R\gamma_2^! Ra_{2*} L'_2 = R\gamma_2^! Ra_{2*} L'_2. \end{aligned}$$

The cohomological correspondence (γ, a_*c') is called the push-forward of (γ', c') by a . It is easy to see that push-forward is compatible with composition. Moreover, we have the following lemma whose proof is also immediate:

Lemma 6.1 *In the above diagram, assume that $X_1 = X'_1$, $X_2 = X'_2$, $a_1 = a_2 = \text{id}$ and a is proper. Let L_i be an object of $D_c^b(X_i, \Lambda)$ for each $i = 1, 2$ and (γ', c') a cohomological correspondence from L_1 to L_2 . Then we have $R\Gamma_c(a_*c') = R\Gamma_c(c')$.*

Next we assume that the right rectangle in the diagram above is cartesian. Let L'_i and (γ', c') be as above. Then we have the cohomological correspondence (γ, a_*c') from $Ra_{1*}L'_1$ to $Ra_{2*}L'_2$ by

$$\gamma_1^* Ra_{1*} L'_1 \xrightarrow{\text{b.c.}} Ra_* \gamma_1'^* L'_1 \xrightarrow{Ra_*(c')} Ra_* R\gamma_2'^! L'_2 \xrightarrow{\text{b.c.}} R\gamma_2^! Ra_{2*} L'_2.$$

On the other hand, let L_i be an object of $D_c^b(X_i, \Lambda)$ for each $i = 1, 2$ and (γ, c) a cohomological correspondence from L_1 to L_2 . Then we have the cohomological correspondence (γ, a^*c) from $a_1^*L_1$ to $a_2^*L_2$ by

$$\gamma_1^* a_1^* L_1 = a^* \gamma_1^* L_1 \xrightarrow{a^*(c)} a^* R\gamma_2^! L_2 \xrightarrow{\text{b.c.}} R\gamma_2^! a_2^* L_2.$$

Finally assume that the left rectangle in the diagram above is cartesian. Let L_i and (γ, c) be as above. Then we have the cohomological correspondence $(\gamma, a^!c)$ from $Ra_1^!L_1$ to $Ra_2^!L_2$ by

$$\gamma_1^* Ra_1^! L_1 \xrightarrow{\text{b.c.}} Ra^! \gamma_1^* L_1 \xrightarrow{Ra^!(c)} Ra^! R\gamma_2^! L_2 = R\gamma_2^! Ra_2^! L_2.$$

These constructions are also compatible with composition.

Next we recall the specialization of cohomological correspondences. Let S be the spectrum of a strict henselian discrete valuation ring on which ℓ is invertible. For an S -scheme X , we denote its special (resp. generic) fiber by X_s (resp. X_η).

Let X_1, X_2 be schemes which are separated of finite type over S and $\gamma: \Gamma \rightarrow X_1 \times_S X_2$ a separated S -morphism of finite type. Let L_i be an object of $D_c^b(X_{i,\eta}, \Lambda)$ for each $i = 1, 2$ and (γ_η, c) a cohomological correspondence from L_1 to L_2 . Then we have the cohomological correspondence $(\gamma_s, R\psi(c))$ from $R\psi L_1$ to $R\psi L_2$ by

$$\gamma_{1,s}^* R\psi L_1 \rightarrow R\psi \gamma_{1,\eta}^* L_1 \xrightarrow{R\psi(c)} R\psi R\gamma_{2,\eta}^! L_2 \rightarrow R\gamma_{2,s}^! R\psi L_2.$$

It is easy to see that this construction is compatible with composition and proper push-forward (cf. [Fuj97, Proposition 1.6.1]).

Now we will give the main result in this section. Let X_i , γ , L_i be as above and Y_i (resp. Z_i) a closed (resp. locally closed) subscheme of $X_{i,s}$. Assume that $\gamma_{1,s}^{-1}(Y_1) = \gamma_{2,s}^{-1}(Y_2)$ and $\gamma_{1,s}^{-1}(Z_1) = \gamma_{2,s}^{-1}(Z_2)$ as subschemes of Γ_s , and denote the former by Γ_Y and the latter by Γ_Z . Then we have the following diagrams whose rectangles are cartesian:

$$\begin{array}{ccc} Y_1 & \xleftarrow{\gamma_{Y,1}} \Gamma_Y & \xrightarrow{\gamma_{Y,2}} Y_2 \\ \downarrow i_1 & & \downarrow i \\ X_{1,s} & \xleftarrow{\gamma_{1,s}} \Gamma_s & \xrightarrow{\gamma_{2,s}} X_{2,s} \end{array} \quad \begin{array}{ccc} Z_1 & \xleftarrow{\gamma_{Z,1}} \Gamma_Z & \xrightarrow{\gamma_{Z,2}} Z_2 \\ \downarrow j_1 & & \downarrow j \\ X_{1,s} & \xleftarrow{\gamma_{1,s}} \Gamma_s & \xrightarrow{\gamma_{2,s}} X_{2,s} \end{array}$$

Therefore, for a cohomological correspondence (γ_η, c) from L_1 to L_2 , the cohomological correspondence $i^*j_*j^!R\psi(c)$ from $i_1^*Rj_{1*}Rj_1^!R\psi L_1$ to $i_2^*Rj_{2*}Rj_2^!R\psi L_2$ is induced. If moreover we assume that γ_1 is proper, then we have

$$R\Gamma_c(i^*j_*j^!R\psi(c)) : R\Gamma_c(X_{1,s}, i_1^*Rj_{1*}Rj_1^!R\psi L_1) \longrightarrow R\Gamma_c(X_{2,s}, i_2^*Rj_{2*}Rj_2^!R\psi L_2).$$

Proposition 6.2 *The morphism $R\Gamma_c(i^*j_*j^!R\psi(c))$ depends only on the cohomological correspondence (γ_η, c) . More precisely, if another S -morphism $\gamma' : \Gamma' \rightarrow X_1 \times_S X_2$ has the same generic fiber as γ and satisfies the conditions that $\gamma'_{1,s}{}^{-1}(Y_1) = \gamma'_{2,s}{}^{-1}(Y_2)$, $\gamma'_{1,s}{}^{-1}(Z_1) = \gamma'_{2,s}{}^{-1}(Z_2)$ and γ'_1 is proper, then the morphism $R\Gamma_c(i'^*j'_*j'^!R\psi(c))$ induced from γ' is equal to $R\Gamma_c(i^*j_*j^!R\psi(c))$ (here i' and j' are defined in the same way as i and j).*

Proof. Since Γ and Γ' have the same generic fiber, there is the “diagonal” in the generic fiber of $\Gamma \times_{X_1 \times_S X_2} \Gamma'$. Let Γ'' be the closure of it in $\Gamma \times_{X_1 \times_S X_2} \Gamma'$. Then Γ'' has the same generic fiber as Γ . We have natural morphisms $\Gamma'' \rightarrow \Gamma$ and $\Gamma'' \rightarrow \Gamma'$, which are proper since γ and γ' are proper. Therefore $\gamma'' : \Gamma'' \rightarrow X_1 \times_S X_2$ also satisfies the same conditions as γ and γ' . By replacing γ' by γ'' , we may assume that there exists a proper morphism $a : \Gamma' \rightarrow \Gamma$ such that $\gamma \circ a = \gamma'$.

Then, it is easy to see that the push-forward of the cohomological correspondence $(\gamma'_s, i'^*j'_*j'^!R\psi(c))$ by a_s coincides with $(\gamma_s, i^*j_*j^!R\psi(c))$. Therefore the proposition follows from Lemma 6.1. \blacksquare

Corollary 6.3 *Let X_1 , X_2 and X_3 be schemes which are separated of finite type over S , Y_i (resp. Z_i) a closed (resp. locally closed) subscheme of X_i , and L_i an object of $D_c^b(X_{i,\eta}, \Lambda)$ for each $i = 1, 2, 3$. Let $\gamma : \Gamma \rightarrow X_1 \times_S X_2$ (resp. $\gamma' : \Gamma' \rightarrow X_2 \times_S X_3$, resp. $\gamma'' : \Gamma'' \rightarrow X_1 \times_S X_3$) be an S -morphism such that γ_1 (resp. γ'_1 , resp. γ''_1) is proper, and (γ_η, c) (resp. (γ'_η, c') , resp. (γ''_η, c'')) a cohomological correspondence from L_1 to L_2 (resp. from L_2 to L_3 , resp. from L_1 to L_3). Moreover we assume that $\gamma_{1,s}^{-1}(Y_1) = \gamma_{2,s}^{-1}(Y_2)$, $\gamma_{1,s}^{-1}(Z_1) = \gamma_{2,s}^{-1}(Z_2)$, $\gamma'_{1,s}{}^{-1}(Y_2) = \gamma'_{2,s}{}^{-1}(Y_3)$, $\gamma'_{1,s}{}^{-1}(Z_2) = \gamma'_{2,s}{}^{-1}(Z_3)$,*

$\gamma_{1,s}^{\prime\prime-1}(Y_1) = \gamma_{2,s}^{\prime\prime-1}(Y_3)$ and $\gamma_{1,s}^{\prime\prime-1}(Z_1) = \gamma_{2,s}^{\prime\prime-1}(Z_3)$. Then, as above, the morphisms

$$\begin{aligned} R\Gamma_c(i^*j_*j^!R\psi(c)) &: R\Gamma_c(X_{1,s}, i_1^*Rj_{1*}Rj_1^!R\psi L_1) \longrightarrow R\Gamma_c(X_{2,s}, i_2^*Rj_{2*}Rj_2^!R\psi L_2), \\ R\Gamma_c(i^*j_*j^!R\psi(c')) &: R\Gamma_c(X_{2,s}, i_2^*Rj_{2*}Rj_2^!R\psi L_2) \longrightarrow R\Gamma_c(X_{3,s}, i_3^*Rj_{3*}Rj_3^!R\psi L_3), \\ R\Gamma_c(i^*j_*j^!R\psi(c'')) &: R\Gamma_c(X_{1,s}, i_1^*Rj_{1*}Rj_1^!R\psi L_1) \longrightarrow R\Gamma_c(X_{3,s}, i_3^*Rj_{3*}Rj_3^!R\psi L_3) \end{aligned}$$

are induced. Assume that the composite of (γ_η, c) and (γ'_η, c') coincides with (γ''_η, c'') . Then we have $R\Gamma_c(i^*j_*j^!R\psi(c')) \circ R\Gamma_c(i^*j_*j^!R\psi(c)) = R\Gamma_c(i^*j_*j^!R\psi(c''))$.

Proof. By Proposition 6.2, we may replace γ'' by $\Gamma \times_{X_2} \Gamma' \longrightarrow X_1 \times_S X_3$. Then the equality is clear, since all the operations for cohomological correspondences are compatible with composition. ■

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